Linköping Studies in Science and Technology

Dissertation No. 498

Algorithms and Complexity for Temporal and Spatial Formalisms

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Linköping 1997

ISBN 91-7219-019-1
ISSN 0345-7524

Abstract

The problem of computing with temporal information was early recognised within the area of artificial intelligence, most notably the temporal *interval algebra* by Allen has become a widely used formalism for representing and computing with qualitative knowledge about relations between temporal intervals. However, the computational properties of the algebra and related formalisms are known to be bad: most problems (like *satisfiability*) are NP-hard. This thesis contributes to finding restrictions (as weak as possible) on Allen's algebra and related temporal formalisms (the *point-interval algebra* and extensions of Allen's algebra for metric time) for which the satisfiability problem can be computed in polynomial time.

Another research area utilising temporal information is that of *reasoning about action*, which treats the problem of drawing conclusions based on the knowledge about actions having been performed at certain time points (this amounts to solving the infamous *frame problem*). One paper of this thesis attacks the computational side of this problem; one that has not been treated in the literature (research in the area has focused on *modelling* only). A nontrivial class of problems for which satisfiability is a polynomial-time problem is isolated, being able to express phenomena such as concurrency, conditional actions and continuous time.

Similar to temporal reasoning is the field of *spatial reasoning*, where spatial instead of temporal objects are the field of study. In two papers, the formalism *RCC-5* for spatial reasoning, very similar to Allen's algebra, is analysed with respect to tractable subclasses, using techniques from temporal reasoning.

Finally, as a spin-off effect from the papers on spatial reasoning, a technique employed therein is used for finding a class of *intuitionistic logic* for which computing inference is tractable.

Acknowledgements

First, I would like to thank my supervisor Christer Bäckström for letting me do this research, and for pointing out to me the existence of the area of temporal reasoning. Next, I have had an enjoyable cooperation with those co-authoring papers of this thesis. Without them, this thesis would have been (if at all) completely different: Peter Jonsson, Marcus Bjäreland and Christer Bäckström.

Many people (other than those above) have contributed to a stimulating research environment with facts, comments and discussions of various kinds, particularly Simin Nadjm-Tehrani, Anders Henriksson, Erik Sandewall, Patrick Doherty, Magnus Andersson, Lars Karlsson, and also anonymous reviewers of papers of the thesis.

Thanks also to Lise-Lott Andersson and Lillemor Wallgren for making administrative issues non-problems, the TUS group for keeping the computers running, and Ivan Rankin for improving my English.

List of Papers

The thesis includes the following eight papers.

I. Peter Jonsson, Thomas Drakengren and Christer Bäckström.
The Computational Complexity of Relating Points with Intervals on the Real Line.

The paper is a revised and extended version of another paper:
Peter Jonsson, Thomas Drakengren and Christer Bäckström.
A Complete Classification of Tractability in the Point-Interval Algebra.
In J. Doyle and L. Aiello, editors, *Proceedings of the 5th International Conference on Principles of Knowledge Representation and Reasoning, KR '96*, pages 352–363, Cambridge, MA, USA October 1996. Morgan Kaufmann.

II. Thomas Drakengren and Peter Jonsson, 1997.
Twenty-one Large Tractable Subclasses of Allen's Algebra.
Artificial Intelligence, 93(1–2):297–319.

The paper is an extended version of another paper:
Thomas Drakengren and Peter Jonsson.
Maximal Tractable Subclasses of Allen's Interval Algebra:
Preliminary Report. In *Proceedings of the 13th (US) National Conference on Artificial Intelligence, AAAI '96*, Portland, OR, USA, August 1996.

III. Thomas Drakengren and Peter Jonsson, 1997.
Eight Maximal Tractable Subclasses of Allen's Algebra with Metric Time.
Journal of Artificial Intelligence Research, 7:25–45.

IV. Thomas Drakengren and Peter Jonsson, 1997.
Towards a Complete Classification of Tractability in Allen's Algebra.
In *Proceedings of the Fifteenth International Joint Conference on Artificial Intelligence, IJCAI '97*, Nagoya, Japan, August 1997.

V. Thomas Drakengren and Marcus Bjäreland.
Reasoning about Action in Polynomial Time.

The paper is a version of the following paper, but with full proofs:
Thomas Drakengren and Marcus Bjäreland, 1997.

Reasoning about Action in Polynomial Time.
In *Proceedings of the Fifteenth International Joint Conference on Artificial Intelligence, IJCAI '97*, Nagoya, Japan, August 1997.

VI. Peter Jonsson and Thomas Drakengren, 1997.
RCC-5 and its Tractable Subclasses.
Journal of Artificial Intelligence Research, 6:211–221.

VII. Peter Jonsson and Thomas Drakengren.
Qualitative Reasoning about Sets Applied to Spatial Reasoning.

VIII. Thomas Drakengren and Peter Jonsson.
Reasoning about Set Constraints Applied to Tractable Inference in Intuitionistic Logic.

1 Thesis Overview

1.1 Topic

The topic of this thesis is the study of computational complexity of temporal and spatial reasoning[1]. Much of the work within the area of Artificial Intelligence (and elsewhere) has been recognised as including the notions of time and space, naturally, since these are fundamental aspects of the real world. It is clear that capabilities of modelling and computing with facts about time and space will be needed for any reasonably flexible system interacting with its environment. The areas of temporal and spatial reasoning in AI have developed from this recognition.

In this thesis, the focus is on *temporal reasoning*, and the results on spatial reasoning can largely be considered as by-products of the research on temporal reasoning. Vila [1994] has written an overview of the general uses of temporal reasoning in AI.

The thesis contributes to the problem of *computing* with temporal and spatial information, as opposed to the task of *modelling* temporal and spatial phenomena. The modelling problem consists of deciding what are the ideal primitive concepts and what is the appropriate formalism to use for a given application. Once it is decided how to model a class of phenomena, methods can be devised for computing facts given information expressed in the chosen formalism. Attacking the problem of computation certainly requires a choice of modelling formalism, and in order for the solutions to the computational problems to be *relevant*, here formalisms that are well established and uncontroversial have been used, at least for the temporal case, where the basis is Allen's interval algebra [Allen, 1983], both in its original form and extended with constructs for expressing metric temporal information. For the spatial case, however, the modelling language used is the spatial algebra, RCC-5 [Bennett, 1994], taken from the RCC approach to spatial reasoning [Randell and Cohn, 1989; Randell *et al.*, 1992b], about which there is still some controversy whether or when it correctly models spatial phenomena [Lemon, 1996].

[1]Instead of *temporal reasoning*, sometimes the expression *temporal constraint reasoning* is used to distinguish the field from that of *reasoning about action*, which deals with the problem of inferring what propositions hold at what time points when it is known that certain actions have been performed (see e.g. [Sandewall, 1994] and paper V of this thesis). I think that this use is unfortunate, since in line with this, any reasoning task having a temporal component could be called *temporal reasoning*.

There are fundamental computational problems associated with temporal and spatial formalisms, including the following:

- Given some information, checking whether some temporal or spatial relation is *entailed* from it, that is, being a logical consequence thereof.

- Checking whether some information is *satisfiable*, that is, checking whether there exists a model that satisfies the information.

Cleary, there is a connection between the two problems, and indeed, they are polynomially equivalent[2] for Allen's algebra [Golumbic and Shamir, 1993], and an analogous proof would easily establish the same result for the RCC-5 algebra. Another reasoning problem that is not as "logical" as the above problems is that of finding a polynomially-sized representation from which models can be extracted in polynomial time [Golumbic and Shamir, 1993].

This thesis is mainly concerned with the *satisfiability* problem. However, it is important to note that although the above two problems are equivalent for the full algebras, this does not necessarily hold for restricted subsets of the algebras (see e.g. the end of the discussion section in paper II).

It was early recognised that computing even with moderately expressive temporal formalisms such as Allen's interval algebra is difficult (the same holds for spatial formalisms); in particular, the polynomial-time algorithm originally proposed by Allen [1983] was designed to be incomplete in order to make the computation reasonably fast. However, this is a very dangerous way of attacking computational obstacles, unless the algorithm is accompanied with some kind of guarantee, like a class for which it is correct. This was not the case for Allen's algorithm.

The computational problems inspired several researchers to apply methods from the field of *complexity theory* to temporal reasoning (and recently, also to spatial reasoning), since the field addresses precisely the correctness of problems related to how difficult they are to compute. Results obtained from this approach include the following:

- It is a polynomial-time problem to check satisfiability (and entailment) for the *point algebra*, i.e. where point variables are related by the relations $<$, $>$, $\leq$, $\geq$, $=$ and $\neq$ [Ladkin and Maddux, 1994].

[2] That is, one problem can be solved in polynomial time using at most a polynomial number of calls to the other, and vice versa.

- Satisfiability for Allen's algebra is NP-complete [Vilain and Kautz, 1986], i.e. it is very unlikely that there exists a polynomial algorithm for the problem.

- Satisfiability for the *point-interval algebra* defined by Vilain [1982], where the only relations are those relating a point with an interval, is NP-complete [Meiri, 1991].

- There exists a *maximal tractable*[3] *subclass* of Allen's algebra, the *ORD-Horn subclass* [Nebel and Bürckert, 1995], i.e. satisfiability is tractable for that subclass, and is tractable for no proper superclass of it. Furthermore, it is the *unique* maximal subclass containing all *basic relations.*

- There exists one maximal tractable subclass in the spatial algebra RCC-5 [Renz, 1996] and one in the spatial algebra RCC-8 [Renz, 1996; Renz and Nebel, 1997].

An important assumption that underlies most of the above results is that variables (taking for example values of temporal intervals or spatial regions) are allowed to be *unrelated*, and this also holds for the results of this thesis. Relaxing that assumption yields more tractable classes [Golumbic and Shamir, 1993].

An area that is related to temporal reasoning is the area of *reasoning about action* (see e.g. [Sandewall, 1994] for an introduction), which is the topic of paper V. In addition to treating relations between temporal entities, *propositions* being true at time points and *actions* which have been executed are also modelled in this field, giving rise to the infamous *frame problem* (the problem of concisely expressing what does *not* change when it is known what *does* change) [McCarthy and Hayes, 1969]. However, emphasis of the research in this area has almost exclusively been on *modelling*, in contrast to the temporal reasoning field. One attempt to address the computational issues of the area is that of Schwalb *et al.* [1994], but it cannot be considered satisfactory, since their "solution" is to sacrifice the completeness requirement of making inferences, but providing no other criterion for correctness[4]. Furthermore, they do not correctly deal with the frame problem.

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[3] The expression "tractable problem" is often used in place of "polynomial-time problem".

[4] Note that any algorithm for entailment that yields no conclusions at all is sound.

1.2 Contributions

This thesis contributes in line with the above results, first by finding new maximal tractable subclasses for temporal and spatial reasoning, thus providing knowledge about the borderline between tractable and intractable reasoning, and second by exhibiting the first tractable class for reasoning about action known in the literature (to my knowledge), and also providing some intractability results in this area. In addition, a new tractable class of *intutionistic logic* is presented; this result was obtained as a by-product of the results on spatial reasoning.

1.3 About the Thesis

The structure of the thesis is simple: this introductory chapter describes the context of and summarises the eight papers, which in turn constitute the rest of the thesis. The papers can be read in any order, but I recommend that papers II and III be read before paper IV, and paper VII before paper VIII.

All of the papers are intended to be reasonably self-contained, presupposing knowledge about discrete mathematics and some theory of computational complexity. (An good introduction to complexity theory is that of Papadimitriou [1994].) Due to this fact, some papers overlap slightly, but this should hopefully not be too disturbing.

Concerning the amount of work put into the papers by the author, it should be noted that the order of the authors reflects this (if indeed it can be quantified), in that the first author of a paper has done most of the research reported therein, and so on for the remaining list of authors. Of course, I am responsible for any errors in the papers.

1.4 Brief Summary of the Papers

Paper I

It is known in the literature that testing satisfiability of a set of point variables over real numbers related with the relations $<$, $>$, $\leq$, $\geq$, $=$ and $\neq$, that is, using the *point algebra* [Vilain, 1982], can be done in polynomial time [Ladkin and Maddux, 1994; van Beek and Cohen, 1990; Gerevini *et al.*, 1993], and that there are several tractable special cases of checking satisfiability using Allen's interval algebra [van Beek, 1989; van Beek, 1990; Golumbic and Shamir, 1993; Nebel and Bürckert, 1995] (the general case is NP-complete [Vilain and Kautz, 1986]). However, not much is known about the computational properties of the *point-interval algebra* [Meiri, 1991], where the only relations allowed are those between a *point* and an *interval*, except for the general satisfiability problem being NP-complete [Meiri, 1991].

Paper I classifies every possible subset of the point-interval algebra into having an NP-complete or polynomial satisfiability problem, respectively. It is proved that there are exactly five maximal tractable subclasses. This represents a step towards obtaining tractable subclasses of Meiri's *qualitative algebra*, which includes the expressive power of the point algebra, the point-interval algebra and the Allen interval algebra in one formalism. An interesting feature of the method of proof employed, is that it requires computer support for testing approximately $2.44 \cdot 10^5$ cases.

Paper II

In Paper I, we could directly attack the problem of finding the complete set of tractable subalgebras of the point-interval algebra. However, this is possible only due to size of the algebra: it contains only 32 relations, making the total number of potential subclasses $2^{32} \approx 4.3 \times 10^9$, which would be theoretically possible to enumerate on a computer. Allen's algebra, on the other hand, contains 8192 relations, yielding $2^{8192} \approx 10^{2466}$ subclasses, clearly being impossible to enumerate on today's computers. Therefore, a less systematic approach to finding tractable subclasses could be excused.

For the Allen algebra, only one maximal tractable class was previously known, the *ORD-Horn* subclass by Nebel and Bürckert [1995], containing 868 relations. This algebra extends all known tractable classes, except for one four-element set found by Golumbic and Shamir [1993]. Paper II expands our knowledge of tractable and maximal tractable subclasses, by defining twenty-one new tractable subclasses whose satisfiability problems are solvable in linear time, all of which are considerably larger than the ORD-Horn class, nine of them being maximal tractable, and the remaining ones nonmaximal (these are extended in paper III, but to the price of using an $O(n^2)$ time algorithm). Each algebra contains exactly three basic relations, and four of them contain the relation $(\prec \succ)$, "before or after", which is needed for expressing the important notion of *sequentiality*. Also the tractable class by Golumbic and Shamir is subsumed by these algebras.

Paper III

This paper combines two directions in temporal reasoning: the search for tractable subalgebras of Allen's algebra, and providing temporal formalisms with the possibility of expressing not only *qualitative* temporal information, but also *metric* information (such as statements like $\frac{3}{4}t - 0.6s \geq 3 \vee x \neq 17$). Eight new subclasses of Allen's algebra are defined, and they are all proved to be maximal tractable. It is proved that all nonmaximal algebras of paper II are included in these algebras. Six of the algebras contain exactly five basic relations each, and two of them contain three basic relations each. Furthermore, we use the new linear-programming approach to temporal reasoning by Jonsson and Bäckström [1996] in order to provide these algebras with metric temporal information: four of the algebras can have metric information on *interval starting points* and the remaining ones on *interval ending points*. It is also proved that we cannot have both, without losing tractability, except for the already known case of *Horn DLRs* [Jonsson and Bäckström, 1996], which provides the ORD-Horn algebra with metric temporal information.

Paper IV

This paper sets papers II and III in perspective, by proving a *partial* classification of tractability in Allen's algebra, in contrast to the *complete* classification of paper I. Given the results of papers II and III, an important question is whether the algebras found are just a few out of a multitude of algebras of comparable expressivity. Using intensive computer support (equivalent to 40 CPU weeks on a Sun SPARC 10 station), we show that the answer to this question has to be answered negatively, in the following precise way: any tractable subclass of Allen's algebra that is not a subset of either the ORD-Horn class or one of the algebras presented in papers II and III, cannot contain more than three basic relations, which have to be $\equiv$, b and the converse of b, where b is one of d, o, s or f. Furthermore, there is no remaining tractable subclass containing the relation $(\prec\ \succ)$, required for expression the property of *sequentiality*. Thus any remaining tractable subclass is necessarily of weak expressivity concerning basic relations and sequentiality. Note that this result also entails a complete classification of tractable algebras containing the basic relation $\prec$.

Paper V

As mentioned in the introduction, the motivation for investigating the problem of reasoning about time in separation has been that the concept of time is used in a large number of artificial intelligence applications. Paper V is the only temporal reasoning paper in this thesis which does not consist *solely* of a temporal component: here we can compute with *propositions* which are true at time points. The paper achieves two goals in this: first, a propositional temporal logic is developed, and second, this logic is extended to model the so-called *common-sense law of inertia*. Logics of this kind are mostly referred to as "logics of action of change", and in modelling power this logic has to be compared to such logics. However, the emphasis of this paper is *not* the traditional one in this field, that of modelling, but that of analysing computational complexity: no analysis of that kind has yet been done in the literature for such logics. It is proved that both in the basic temporal logic and in the extension for reasoning about action, the satisfiability problems are NP-complete, but perhaps more important, nontrivial tractable classes are found in both, capable of modelling phenomena like concurrent actions, continuous time and conditional results of actions. The expressivity of the temporal component is equal to that of Horn DLRs [Jonsson and Bäckström, 1996], which means that qualitative as well as metric time can be used.

One interesting (although not surprising) conclusion to draw from this paper is that sometimes a separation of temporal and non-temporal components is not the best way to attack the computational problems involving time: we obtain the tractable cases by *coding* the logics into Horn DLRs, and not by treating truth-value propositions and temporal propositions separately. It is not at all clear that a separated approach could handle this as efficiently.

Paper VI

In this paper, we apply methods from results in temporal reasoning on a related field, that of *spatial* reasoning. Also in this field, formalisms like Allen's algebra, the point-interval algebra and the point algebra have been invented to model qualitative spatial phenomena. In this paper, we take an existing such formalism from the RCC approach to spatial reasoning [Randell and Cohn, 1989; Randell *et al.*, 1992b], in particular the RCC-5 algebra [Bennett, 1994], and classify every subset of it (there are 32 relations in it, so the effort is comparable to that of paper I) into having a tractable or NP-complete satisfiability problem. Only one maximal tractable subclass of this algebra was known previously [Renz, 1996], which in turn built on earlier results by Nebel [1995]. In the paper we find the remaining three maximal tractable classes.

Paper VII

This paper relates the temporal and spatial reasoning areas: generalising the method of proof employed by Jonsson and Bäckström [1996] for proving that satisfiability for Horn-DLRs is tractable, we prove that for a corresponding Horn class of *reasoning about nonempty sets*, satisfiability checking can be done in polynomial time. Then we apply this result to finding a simpler algorithm for the previously known maximal tractable class of the RCC-5 algebra for spatial reasoning [Renz, 1996]. The existing polynomial algorithm involves complicated transformations using modal logic, whereas our algorithm is based on simple set-theoretical manipulations. In addition, by approaching the problem from the more abstract set-theoretical level, it is possible to express statements about spatial relationships that are not expressible in the RCC-5 algebra alone. Furthermore, some topological properties of a tractable subclass of the RCC-5 algebra are investigated, showing that satisfiability of that subclass is equivalent to satisfiability in three-dimensional space. This complements the results by Grigni *et al.* [1995], which show that satisfiability in two-dimensional space for a restricted version of the same subclass is NP-complete.

Paper VIII

This paper is somewhat more loosely related to the central topic of the thesis, that of reasoning about temporal and spatial relations, than the rest of the papers. It connects mainly to paper VII, by providing a new tractable class of tractable reasoning about sets. The difference from paper VII is that we investigate the case when empty sets are also allowed as set variables, and it turns out that the Horn class found in paper VII has an NP-complete satisfiability problem under this assumption. A new Horn-like class (being a subset of the class of paper VII) is further defined, which is proved to have a tractable satisfiability problem also when empty sets are allowed. The technique for proving tractability is very similar to that of paper VII, and was originally employed by Jonsson and Bäckström [1996].

Using a connection between topological spaces and *intuitionistic logic* established by Tarski [1956], we use the tractable class to obtain a subclass of intuitionistic logic for which inference is tractable. This result is somewhat related to spatial reasoning: in a more expressive approach to spatial reasoning than the RCC-5 algebra, the RCC-8 algebra [Bennett, 1994], a different tractable class of intuitionistic logic was used in order to find a tractable class [Nebel, 1995] (extended by Renz and Nebel [1997]), and it is possible that this class can be used in a similar way, to find new tractable subclasses of the RCC-8 algebra.

2 The Papers in Context

This section describes a broader perspective of the papers than the one given in Section 1.4.

2.1 Papers I–IV: Temporal Reasoning

Outside the area of Artificial intelligence, much work has been done concerning the *modelling* of temporal phenomena using various kinds of temporal logics (see e.g. Gabbay *et al.* [1994] for an overview). However, the *computational* issues dealt with are usually on the level of establishing *decidability* or *axiomatisability*, due to the expressiveness of the logics involved. Therefore, since the emphasis of this thesis is on finding polynomial-time algorithms for reasoning about time, these logics merit less attention here.

Within the area of Artificial Intelligence, early attempts at computing with temporal information are those of Bruce [1972] and Kahn and Gorry [1977]. The former seems to be the first one defining what constitute the basic relations of Allen's algebra. Both of these present an implemented system for reasoning about temporal information, but there is no evidence of the existence of correctness proofs for their algorithms (which are not even reported), and at least for the latter, the intended semantics for the formalism employed seems to be unclear.

Allen [1983] seems to be one of the first to take formalisation, computation and correctness seriously, by giving a precise definition of his algebra of interval relations, providing an algorithm for computing the closure of an interval network (that is, finding the strongest possible implied relation between two interval variables), and showing *incompleteness* of the proposed algorithm. In Allen's paper, the reason for emphasising the *algebraic operations* (making it an algebra) on interval relations (*composition*, *intersection* and *converse* of relations) was primarily for the operation of his particular closure algorithm. Interestingly, the property of being an algebra was later exploited for a completely different purpose, pioneered by Nebel and Bürckert [1995] and frequently used in papers I, II, III, IV and VI of this thesis: it turns out that the satisfiability problem for interval networks using only a restricted set S of interval relations is NP-complete if and only if the satisfiability problem using the set S' of interval relations is, where S' is the least set of relations containing S and being closed under the algebraic operations (this is a different kind of *closure*).

Soon, the computational properties of Allen's algebra were investigated by Vilain and Kautz [1986]: the satisfiability problem is NP-complete, and so is the problem of computing the closure of an interval network. This means that it is very unlikely that there exist polynomial-time (let alone efficient) algorithms for these problems. Essentially three lines of attack have been pursued in response to this fact:

- Not solving the intended problem, by designing the algorithms as being incomplete and hoping that they will work for most cases anyway. Allen's original algorithm [Allen, 1983] is an example of this approach.

- Trying to design fast algorithms for the full algebra, concentrating on the performance on certain randomly-generated instances and on benchmark problems which are naturally occurring in, for instance, molecular biology [Valdés-Pérez, 1987; Ladkin and Reinefeld, 1992; van Beek and Manchak, 1996; Mitra and Loganantharaj, 1996; Nebel, 1996]. Of course, these algorithms cannot solve the full problem efficiently, but it appears that improvement is made towards solving instances which are of somewhat practical size. However, the difficulty of empirically analysing algorithms (see e.g. [Johnson, 1996]) will probably require more investigation.

- Restricting the set of relations, proving that the problem for the restricted set of relations is a polynomial-time one. This is a more theoretical approach than the former, since no experimental data is required for establishing results. It can also be argued that it might be appropriate to investigate what efficient algorithms exist before a general-purpose superpolynomial algorithm is employed.

The papers of this thesis follow the third approach, which was indeed the first one to be pursued: Vilain and Kautz [1986] found that when the *point* *algebra* [Vilain, 1982] is used (that is, relating time point variables with the relations $<$, $\leq$, $=$, $\geq$, $>$ and $\neq$), satisfiability can be solved in polynomial time (the algorithm was improved by Gerevini *et al.* [1993] to a linear-time one), and expressing interval relations in terms of these yields a fragment of Allen's algebra for which computing satisfiability is polynomial; the *pointisable subalgebra*. However, it was not clear whether this subalgebra could be extended further, until Nebel and Bürckert [1995] defined a superset thereof, the *ORD-Horn* algebra, which furthermore was shown to be maximal tractable, that is, impossible to extend without losing tractability, and also that no other subalgebras than those contained in the ORD-Horn algebra can both be tractable and contain all the basic relations. Also Golumbic and Shamir [1993] defined several tractable subclasses of Allen's algebra (and provided NP-completeness results) using graph-theoretic methods. One of their tractable classes is not included in the ORD-Horn class (there are also other tractable classes, but these cannot express the relation "unrelated", as discussed above).

When time intervals or points are related using Allen's algebra or the point-algebra, only *qualitative* information can be expressed, that is, we cannot express, for instance, that a time point comes five seconds after some other time point. In order to remedy this, several different formalisms allowing such expressiveness have been developed and analysed [Dechter *et al.*, 1991; Meiri, 1991; Kautz and Ladkin, 1991; Koubarakis, 1992; Lassez and McAloon, 1992]. By the main result of Jonsson and Bäckström [1996], all tractable classes defined in these papers (except for two by Meiri, involving nonstandard entities such as discrete domains and multi-intervals) are unified into a single tractable formalism based on linear programming, that of *Horn-DLRs*, which also includes the ORD-Horn algebra (this result was independently found by Koubarakis [1996], but using a less transparent algorithm). The contributions of papers I–IV fall mainly within this category of results. Related to papers I and IV is also the complete classification by Pe'er and Shamir [1997] of tractable subclasses of a *subset* of Allen's algebra (using the same kind of *macro relations* as Golumbic and Shamir [1993]) with a nonstandard semantics where all intervals are required to have the same length.

We should also mention that much research is being done on finding special-purpose data structures for making reasoning practically efficient even though a polynomial-time algorithm is already known (note that the requirement of polynomial-time algorithms is not a sufficient condition for practical usefulness), mostly for the point-algebra [Ghallab and Alaoui, 1989;

Dean and McDermott, 1987; Dorn, 1992; Schrag *et al.*, 1992; Delgrande and Gupta, 1996].

2.2 Paper V: Reasoning about Action

Early in the history of Artificial Intelligence, McCarthy [1958] introduced logic as a means for representing knowledge. Soon, several problems were discovered with this approach; except for the inefficiencies involved with running a theorem prover for computing consequences of facts [Green, 1969], the representation issues themselves provided some problems, notably the infamous *frame problem*, identified by McCarthy and Hayes [1969]. The frame problem is how to represent what does *not* change in the world, when it is known that certain actions affecting the world have been performed, and maybe some additional observations are to be taken into account. Since the emphasis of this thesis is not on modelling but on computing, it serves no purpose to describe the development of these modelling formalisms, but it is interesting to note that after several papers proposing solutions to the frame problem, Hanks and McDermott [1986] pointed out that none of these was correct, using the *Yale shooting problem* (thereafter a prototypical example for reasoning about action) as a diagnostic example. As a reaction to this, some researchers fortunately started to be more systematic [Sandewall, 1994; Gelfond and Lifschitz, 1993; Kartha and Lifschitz, 1994; Lin and Shoham, 1991], by starting to define clearly what problem was to be solved, and under which assumptions the approach would hold as a correct solution to the frame problem. This so-called *systematic approach* (as opposed to the previously used *example-based approach*) is followed in paper V, of course, by stating mathematically what problem is to be solved and proving that this is indeed done.

In this area, the focus of interest has generally been that of *modelling* rather than computation, probably by the impact of the complications surrounding the frame problem, and by the fact that formalisms employed are almost always at least as expressive as first order logic, where computing entailment is known to be undecidable [Boolos and Jeffrey, 1989]. To our knowledge, no complexity analyses of such formalisms have yet appeared in the literature[5].

[5] Liberatore [1997] has performed a complexity analysis of the language $\mathcal{A}$ (which is less expressive than that used in the paper) for reasoning about action [Gelfond and Lifschitz, 1993]; the publication is however unrefereed.

2.3 Papers VI-VII: Spatial Reasoning

The problem of spatial reasoning is slightly more involved than that of temporal reasoning: whereas the choice of ontological entities in temporal reasoning is relatively obvious (points or intervals; sometimes perhaps multi-intervals or sets of time points [Meiri, 1991; Ladkin, 1986a; Ladkin, 1986b]), this is not the case for spatial reasoning. For instance, we have the following aspects to decide on: How many *dimensions* (two or three[6]) are to be used? Is the *form* or *orientation* of objects to be represented? How can (qualitative) spatial relations be naturally chosen? For a discussion of such issues, see Freksa and Röhrig [1993].

In this thesis we shall only consider spatial reasoning in its form closest related to temporal reasoning formalisms like Allen's algebra, and not the more general (and more imprecise) kind of reasoning as, for instance, *naïve physics* [Hayes, 1979; Hayes, 1985], which concerns representing knowledge about physical processes without using standard physics.

The so-called RCC approach by Randell, Cohn and Cui [Randell and Cohn, 1989; Randell *et al.*, 1992b; Bennett, 1994] is a first-order formalism for expressing knowledge about spatial relationships, with two limited sub-formalisms, named RCC-5 and RCC-8. These formalisms are defined as algebras like the Allen interval algebra, containing five and eight basic relations, respectively, but of different expressive power. Essentially, RCC-5 expresses relations between nonempty sets, whereas RCC-8 instead has sets in a topological space as its ontological primitive, thus distinguishing the border of a region from the rest of it. Until recently, the computational properties of this approach were largely unknown; existing methods were either not known to be complete [Egenhofer, 1991; Hernàndez, 1994] or very inefficient [Randell *et al.*, 1992b; Bennett, 1994]. Even computing a transitivity table posed a computational problem [Randell *et al.*, 1992a]! Nebel [1995] presented the first results on the computational complexity of these formalisms, showing that consistency-checking of the basic relations of RCC-8 can be done in polynomial time, and Grigni *et al.* [1995] classified the complexity of several cases when models are required to be two-dimensional. Renz [1996] extended Nebel's result, providing one maximal tractable subclass of RCC-8 (see also Renz and Nebel [1997]) and also one of RCC-5. Paper VI identifies the remaining tractable subclasses of RCC-5, and paper VII presents an algorithm for reasoning about arbitrary nonempty sets, yielding as a side-effect a new, simpler algorithm for the maximal tractable RCC-5 subclass of Renz [1996].

[6]One-dimensional spatial reasoning seems to collapse to temporal reasoning.

Furthermore, we cast some light on the semantical foundations of RCC-5 by showing that for a particular subclass, satisfiability is equivalent to satisfiability in $\mathbb{R}^3$ (see Lemon [1996] for a criticism of the semantics of various spatial logics).

3 Paper VIII: Intuitionistic Logic

Intuitionistic logic represents an attempt to formalise the ideas arising from Brouwer's criticism of the foundations of mathematics [Heyting, 1971; Dragalin, 1988]. Not much is known about the computational properties thereof, except for inference being PSPACE-complete in the propositional case [Statman, 1979] (there also exists an $O(nlogn)$-space procedure [Hudelmaier, 1993]), and the tractable class used by Nebel [1995] to find a tractable class of the RCC-8 algebra for spatial reasoning. In paper VIII, we improve on this by finding a new tractable subclass of intuitionistic logic, using a technique for reasoning about sets similar to that of paper VII. Furthermore, the results on reasoning about sets are interesting on their own, as are the corresponding results of paper VII.

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Paper I

The Computational Complexity of Relating Points with Intervals on the Real Line

Peter Jonsson, Thomas Drakengren and Christer Bäckström

ABSTRACT

Vilain has suggested the point-interval algebra for reasoning about the relative positions of points and intervals on the real line. The satisfiability problem for the full point-interval algebra is known to be NP-complete but the computational complexity of its $4,294,967,296$ subclasses has not been investigated. We classify these subclasses with respect to whether their corresponding satisfiability problems are polynomial-time or NP-complete. The classification reveals that there are exactly five maximal subclasses having a polynomial-time satisfiability problem.

1 Introduction

Problems concerned with the realizability of a set of points or intervals (along the real line) subject to constraints have been a very active research area for some time. Such problems arise in several application areas including operations research [Papadimitriou and Yannakakis, 1979], combinatorics [Roberts, 1976], planning [Allen, 1991], natural language processing [Song and Cohen, 1988] and temporal reasoning such as time serialization in archeology [Golumbic and Shamir, 1993], The archetypical example is *Allen's interval algebra* [Allen, 1983] which is based on relations between pairs of intervals on the real line. Intervals are related by disjunctions of 13 basic relations which yields 8192 possible relations. Given a set of interval variables related by Allen relations, the satisfiability problem is to decide whether there exists an interval interpretation of the variables satisfying the relations or not. This problem (which is NP-complete [Vilain and Kautz, 1986]) has been studied in a large number of papers [Vilain *et al.*, 1989; Golumbic and Shamir, 1993; Nebel and Bürckert, 1995; Drakengren and Jonsson, 1997c; Drakengren and Jonsson, 1997a; Drakengren and Jonsson, 1997b]. A variant of Allen's algebra where all intervals are required to have unit length is studied in [Pe'er and Shamir, 1997].

Another algebra which has been thoroughly investigated is the *point algebra* [Vilain, 1982]. This algebra considers relations between pairs of points on the real line. In this case, we have three basic relations and eight disjunctive relations. The satisfiability problem for the point-interval algebra is defined analogously to the satisfiability problem for Allen's algebra. However, this problem is solvable in polynomial time [Vilain *et al.*, 1989].

As a natural intermediate problem, the *point-interval algebra* has been suggested [Vilain, 1982]. Here, a point p and an interval I on the real line can be related in five mutually exclusive ways depicted in Table 1. The satisfiability problem for the full point-interval algebra is known to be NP-complete [Meiri, 1996]. One way of dealing with the complexity of the point-interval algebra is to restrict the algebra only to contain subsets of its 32 relations and study the complexity of these subclasses. This approach has been quite successful in the case of Allen's algebra where a large number of polynomial-time solvable subclasses has been reported [van Beek and Cohen, 1990; Golumbic and Shamir, 1993; Nebel and Bürckert, 1995; Drakengren and Jonsson, 1997a; Drakengren and Jonsson, 1997c]. The main result of this article is a complete classification of all subclasses of the point-interval algebra with respect to the computational complexity of the corresponding satisfiability problem. Our study reveals that there exist five maximal subclasses whose satisfiability problem is solvable in polynomial time. Furthermore, if a subclass is not a subset of one of these subclasses then the satisfiability problem is NP-complete. Similar dichotomy results have been shown for the generalized satisfiability problem [Schaefer, 1978] and the directed subgraph homeomorphism problem [Fortune *et al.*, 1980], just to mention a few.

A few words on methodology are appropriate at this point. The proof of the main theorem relies on a quite extensive case analysis performed by a computer. The number of cases considered in this analysis was approximately 2.4×10^5. Naturally, such an analysis cannot be reproduced in an article or be verified manually. To allow for the verification of our results, we include a description of the program used in the analysis. The program can be obtained from the authors.

The rest of this article is organized as follows: Section 2 defines the point-interval algebra and some auxiliary concepts. Section 3 contains the classification of subclasses and Section 4 concludes the article. Most of the proofs are postponed to the appendix. This article is an extended and corrected version of an earlier conference paper [Jonsson *et al.*, 1996].

2 Point-Interval Relations

The point-interval algebra is based on the notions of *points*, *intervals* and *binary relations* on these. A point p is a variable interpreted over the set of real numbers $\mathbb{R}$. An interval I is represented by a pair $\langle I^-, I^+\rangle$ satisfying $I^- < I^+$ where I^- and I^+ are interpreted over $\mathbb{R}$. We assume that we have a fixed universe of variable names for points and intervals. Then, a $\mathcal{V}$*-interpretation* is a function $\Im$ that maps point variables to $\mathbb{R}$ and interval variables to $\mathbb{R} \times \mathbb{R}$ and satisfies the previously stated restrictions. We will frequently extend the notation by denoting the first component of $\Im(I)$ by $\Im(I^-)$ and the second by $\Im(I^+)$.

Given an interpreted point and an interpreted interval, their relative positions can be described by exactly one of the elements of the set $\mathbf{B}$ of five *basic point-interval relations* where each basic relation can be defined in terms of its endpoint relations (see Table 1). A formula of the form pBI where p is a point, I an interval and $B \in \mathbf{B}$, is said to be satisfied by a $\mathcal{V}$-interpretation iff the interpretation of the points and intervals satisfies the endpoint relations specified in Table 1.

To express indefinite information, unions of the basic relations are used which lead to 2^5 distinct binary *point-interval relations*. Naturally, a set of basic relations is to be interpreted as a disjunction of its member relations. A point-interval relation is written as a list of its members, *e.g.*, ($\texttt{b d a}$). The set of all point-interval relations $2^{\mathbf{B}}$ is denoted by $\mathcal{V}$. Relations of special interest are the *null* relation $\varnothing$ (also denoted by $\perp$) and the *universal* relation $\mathbf{B}$ (also denoted $\top$).

A formula of the form $p(B_1, \ldots, B_n)I$ is called a *point-interval* formula. Such a formula is satisfied by a $\mathcal{V}$-interpretation $\Im$ iff pB_iI is satisfied by $\Im$ for some i, $1 \leq i \leq n$. A set Θ of point-interval formulae is said to be $\mathcal{V}$-*satisfiable* iff there exists an $\mathcal{V}$-interpretation $\Im$ that satisfies every formula of Θ. Such a satisfying $\mathcal{V}$-interpretation is called a $\mathcal{V}$*-model* of Θ. The reasoning problem we will study is the following:

INSTANCE: A finite set Θ of point-interval formulae.
QUESTION: Does there exist a $\mathcal{V}$-model of Θ?

We denote this problem $\mathcal{V}$-SAT. In the following, we often consider restricted versions of $\mathcal{V}$-SAT where relations used in the formulae in Θ are only from a subset $\mathcal{S}$ of $\mathcal{V}$. In this case we say that Θ is a set of formulae over $\mathcal{S}$ and use a parameter in the problem description to denote the subclass under consideration, *e.g.* $\mathcal{V}$-SAT($\mathcal{S}$).

Meiri's extended definition of the point-interval algebra consists of $\mathcal{V}$ equipped with the two binary operations *intersection* and *composition*. However, such a definition does not constitute an algebra because it is not closed under composition. We replace the composition operation with an operation on $\mathcal{V}$ which we will refer to as *3-composition*.

Definition 2.1 Let $\mathbf{B} = \{\texttt{b}, \texttt{s}, \texttt{d}, \texttt{f}, \texttt{a}\}$. The *point-interval algebra* consists of the set $\mathcal{M} = 2^{\mathbf{B}}$ and the operations binary *intersection* (denoted by $\cap$) and ternary *3-composition* (denoted by $\otimes$). They are defined as follows:

$$\forall p, I : p(R \cap S)I \Leftrightarrow pRI \wedge pSI;$$

$$\forall p, I : p(R \otimes S \otimes T)I \Leftrightarrow \exists q, J : (qRI \wedge qSJ \wedge pTJ).$$

It can easily be verified that

$$R \otimes S \otimes T = \bigcup \{B \otimes B' \otimes B'' | B \in R, B' \in S, B'' \in T\},$$

Basic relation	Symbol	Example	Endpoint relation
p before I	`b`	`p` `III`	$p < I^-$
p starts I	`s`	`p` `III`	$p = I^-$
p during I	`d`	`p` `III`	$I^- < p < I^+$
p finishes I	`f`	`p` `III`	$p = I^+$
p after I	`a`	`p` `III`	$p > I^+$

Table 1: The five basic relations of the $\mathcal{V}$-algebra. The endpoint relation $I^- < I^+$ that is required for all relations has been omitted.

i.e., 3-composition is the union of the component-wise 3-composition of basic relations. In Table 2, we present the tables for 3-composition of basic relations. We can see that, for instance, $(\mathtt{f}) \otimes (\mathtt{b}) \otimes (\mathtt{s}) = (\mathtt{a})$.

Next, we introduce a *closure* operation $\mathcal{C}_\mathcal{V}$. This operation will simplify some of the following proofs.

Definition 2.2 Let $\mathcal{S} \subseteq \mathcal{M}$. Then we denote by $\mathcal{C}_\mathcal{V}(\mathcal{S})$ the $\mathcal{V}$-*closure* of $\mathcal{S}$, defined as the least subset of $\mathcal{M}$ containing $\mathcal{S}$ which is closed under intersection and 3-composition.

Given a set $\mathcal{S} \subseteq \mathcal{M}$, we can easily compute $\mathcal{C}_\mathcal{V}(\mathcal{S})$. Let $\Psi : 2^\mathcal{M} \to 2^\mathcal{M}$ be defined as

$$\Psi(X) = X \cup \{x \cap y \mid x, y \in X\} \cup \{x \otimes y \otimes z \mid x, y, z \in X\}.$$

Obviously, Ψ is a monotone function on the complete lattice $\langle 2^\mathcal{M}, \subseteq \rangle$. Since $\Psi^i(\mathcal{S}) \subseteq \Psi^{i+1}(\mathcal{S})$ for all i and $|\mathcal{M}| = 32$, there exists a $k \leq 32$ such that $\Psi^k(\mathcal{S}) = \Psi^{k+1}(\mathcal{S})$. Clearly, $\Psi^k(\mathcal{S}) = \mathcal{C}_\mathcal{V}(\mathcal{S})$. A program for computing $\mathcal{V}$-closures can be obtained from the authors.

The proof of the following result is omitted since proofs of analogous results can be found in [Nebel and Bürckert, 1995].

Lemma 2.3 Let $\mathcal{S} \subseteq \mathcal{V}$. $\mathcal{V}$-SAT$(\mathcal{S})$ is in P iff $\mathcal{V}$-SAT$(\mathcal{C}_\mathcal{V}(\mathcal{S}))$ is in P. $\mathcal{V}$-SAT$(\mathcal{S})$ is NP-hard iff $\mathcal{V}$-SAT$(\mathcal{C}_\mathcal{V}(\mathcal{S}))$ is NP-hard.

We continue by defining a duality operation on $\mathcal{V}$.

$R =$ b					
	b	s	d	f	a
b	(b s d f a)	(b s d f a)	(b s d f a)	(b s d f a)	(b s d f a)
s	(b)	(b)	(b s d f a)	(b s d f a)	(b s d f a)
d	(b)	(b)	(b s d f a)	(b s d f a)	(b s d f a)
f	(b)	(b)	(b)	(b)	(b s d f a)
a	(b)	(b)	(b)	(b)	(b s d f a)

$R =$ s					
	b	s	d	f	a
b	(b s d f a)	(d f a)	(d f a)	(d f a)	(d f a)
s	(b)	(s)	(d f a)	(d f a)	(d f a)
d	(b)	(b)	(b s d f a)	(d f a)	(d f a)
f	(b)	(b)	(b)	(s)	(d f a)
a	(b)	(b)	(b)	(b)	(b s d f a)

$R =$ d					
	b	s	d	f	a
b	(b s d f a)	(d f a)	(d f a)	(d f a)	(d f a)
s	(b s d)	(d)	(d f a)	(d f a)	(d f a)
d	(b s d)	(b s d)	(b s d f a)	(d f a)	(d f a)
f	(b s d)	(b s d)	(b s d)	(d)	(d f a)
a	(b s d)	(b s d)	(b s d)	(b s d)	(b s d f a)

$R =$ f					
	b	s	d	f	a
b	(b s d f a)	(a)	(a)	(a)	(a)
s	(b s d)	(f)	(a)	(a)	(a)
d	(b s d)	(b s d)	(b s d f a)	(a)	(a)
f	(b s d)	(b s d)	(b s d)	(f)	(a)
a	(b s d)	(b s d)	(b s d)	(b s d)	(b s d f a)

$R =$ a					
	b	s	d	f	a
b	(b s d f a)	(a)	(a)	(a)	(a)
s	(b s d f a)	(a)	(a)	(a)	(a)
d	(b s d f a)	(b s d f a)	(b s d f a)	(a)	(a)
f	(b s d f a)	(b s d f a)	(b s d f a)	(a)	(a)
a	(b s d f a)	(b s d f a)	(b s d f a)	(b s d f a)	(b s d f a)

Table 2: 3-composition in the $\mathcal{V}$-algebra. Each subtable represents one of the five possible choices of the R relation. Within each subtable, the vertical axis represents the S relation and the horizontal axis the T relation.

Definition 2.4 Let $R \in \mathcal{V}$. Define $\mathcal{D}_{\mathcal{V}}(R)$ as the set $\{\beta(r) \mid r \in R\}$ where $\beta(r)$ is defined as follows:

- $\beta(\mathtt{b}) = \mathtt{a}$;
- $\beta(\mathtt{s}) = \mathtt{f}$;
- $\beta(\mathtt{d}) = \mathtt{d}$;
- $\beta(\mathtt{f}) = \mathtt{s}$;
- $\beta(\mathtt{a}) = \mathtt{b}$.

Let $\mathcal{S} \subseteq \mathcal{V}$. Define $\mathcal{D}_{\mathcal{V}}(\mathcal{S})$ as the set $\{\mathcal{D}_{\mathcal{V}}(R) \mid R \in \mathcal{S}\}$.

Lemma 2.5 Let $\mathcal{S} \subseteq \mathcal{V}$. There is a polynomial-time reduction from $\mathcal{V}$-SAT$(\mathcal{D}_{\mathcal{V}}(\mathcal{S}))$ to to $\mathcal{V}$-SAT$(\mathcal{S})$ and vice versa.

Proof: We show the reduction from $\mathcal{V}$-SAT$(\mathcal{D}_{\mathcal{V}}(\mathcal{S}))$ to $\mathcal{V}$-SAT$(\mathcal{S})$; the other reduction is analogous. Let Θ be an arbitrary instance of the $\mathcal{V}$-SAT$(\mathcal{D}_{\mathcal{V}}\mathcal{S})$ problem. Let $\Theta' = \{p\mathcal{D}_{\mathcal{V}}(R)I \mid pRI \in \Theta\}$. Obviously, Θ' is an instance of the $\mathcal{V}$-SAT(S) problem. Assume Θ has a $\mathcal{V}$-model $\Im$. Construct a new $\mathcal{V}$-interpretation $\Im'$ as follows:

> $\Im'(p) = -\Im(p)$ for each point p appearing in Θ;
>
> $\Im'(I^-) = -\Im(I^+)$ and $\Im'(I^+) = -\Im(I^-)$ for each interval I appearing in Θ.

It is not hard to see that $\Im'$ is a $\mathcal{V}$-model of Θ'. Showing the converse direction is analogous. $\Box$

Corollary 2.6 Let $\mathcal{S} \subseteq \mathcal{V}$. $\mathcal{V}$-SAT$(\mathcal{S})$ is in P iff $\mathcal{V}$-SAT$(\mathcal{D}_{\mathcal{V}}(\mathcal{S}))$ is in P. $\mathcal{V}$-SAT$(\mathcal{S})$ is NP-hard iff $\mathcal{V}$-SAT$(\mathcal{D}_{\mathcal{V}}(\mathcal{S}))$ is NP-hard.

3 Classification of $\mathcal{V}$

We begin this section by defining five subalgebras of the point-interval algebra having a polynomial-time $\mathcal{V}$-SAT problem. Later on, we show that these algebras are the only maximal subalgebras of $\mathcal{V}$ with this property. Before we can define the algebras, we need a definition concerning the point algebra.

Definition 3.1 A *point algebra (PA) formula* is an expression of the form xry where r is a member of $\{<, \leq, =, \neq, \geq, >, \perp, \top\}$ and x, y denote real-valued variables. The symbols $<, \leq, =, \neq, \geq, >$ denote the relations "strictly less than", "less than" and so on. The symbol $\perp$ denotes the relation $\varnothing$ which is unsatisfiable for every choice of $x, y \in \mathbb{R}$ and $\top$ denotes the relation $\mathbb{R} \times \mathbb{R}$ which is satisfiable for every choice of $x, y \in \mathbb{R}$.

Let Ω be a set of PA formulae and X the set of variables appearing in Ω. An assignment of real values to the variables in X is said to be a *PA-interpretation* of Ω. Furthermore, Ω is *PA-satisfiable* iff there exists a PA-interpretation $\Im$ such that for each formula $xry \in \Omega$, $\Im(x) r \Im(y)$ holds. Such a PA-interpretation $\Im$ is said to be a *PA-model* of Ω.

The first point-interval subalgebra we will consider has a very close connection to PA.

Definition 3.2 The set $\mathcal{V}^{23}$ consists of those relations in $\mathcal{V}$ that can be expressed as one or more PA formulae over points and endpoints of intervals.

Alternatively, $\mathcal{V}^{23}$ can be characterized in the following two ways:

$$\mathcal{V}^{23} = \mathcal{C}_{\mathcal{V}}(\{(\mathtt{s}), (\mathtt{f\ a}), (\mathtt{b\ d\ f})\}); \text{ or}$$

$$\mathcal{V}^{23} = \{r \in \mathcal{V} \mid (\mathtt{d}) \subseteq r \text{ or } r \subseteq (\mathtt{b\ s}) \text{ or } r \subseteq (\mathtt{f\ a})\}.$$

The other four subalgebras are defined with the aid of $\mathcal{C}_{\mathcal{V}}$ and $\mathcal{D}_{\mathcal{V}}$.

Definition 3.3

$$v^{20}_{\mathtt{s}} = \{(\mathtt{s}), (\mathtt{b\ s}), (\mathtt{b\ a}), (\mathtt{b\ s\ f\ a}), (\mathtt{b\ s\ d\ a})\},$$

$$\mathcal{V}^{20}_{\mathtt{s}} = \mathcal{C}_{\mathcal{V}}(v^{20}_{\mathtt{s}})$$

$$\mathcal{V}^{20}_{\mathtt{f}} = \mathcal{D}_{\mathcal{V}}(\mathcal{V}^{20}_{\mathtt{s}})$$

$$\mathcal{V}^{17}_{\mathtt{s}} = \{\perp\} \cup \{r \in \mathcal{V} \mid (\mathtt{s}) \subseteq r\}$$

$$\mathcal{V}^{17}_{\mathtt{f}} = \mathcal{D}_{\mathcal{V}}(\mathcal{V}^{17}_{\mathtt{s}})$$

Given a subalgebra $\mathcal{V}^x_y$, x denotes the number of relations in the algebra. Let $\mathcal{V}_P$ be the set of all subalgebras in Definition 3.3 together with the algebra $\mathcal{V}^{23}$. The relations included in each of these algebras can be found in Table 3. Further, let $\mathcal{V}_{NP}$ denote the set of subalgebras listed in Table 4. We have the following theorem.

Theorem 3.4 If $V \in \mathcal{V}_P$ then $\mathcal{V}$-SAT(V) is in P. If $V \in \mathcal{V}_{NP}$ then $\mathcal{V}$-SAT(V) is NP-hard.

Proof: See Appendices A and B for the results concerning $\mathcal{V}_P$ and $\mathcal{V}_{NP}$, respectively. $\square$

The main theorem can now be stated.

Theorem 3.5 For $\mathcal{S} \subseteq \mathcal{V}$, $\mathcal{V}$-SAT($\mathcal{S}$) is in P iff $\mathcal{S}$ is a subset of some member of $\mathcal{V}_P$. Otherwise, $\mathcal{V}$-SAT($\mathcal{S}$) is NP-complete.

Proof: *if:* For each $C \in \mathcal{V}_P$, $\mathcal{V}$-SAT(C) is in P by Theorem 3.4.
only-if: Choose $\mathcal{S} \subseteq \mathcal{V}$ such that $\mathcal{S}$ is not a subset of any algebra in $\mathcal{V}_P$. For each subalgebra C in $\mathcal{V}_P$, choose a relation x such that $x \in \mathcal{S}$ and $x \notin C$. This can always be done since $\mathcal{S} \not\subseteq C$. Let X be the set of these relations. We make three observations about X:

1. $|X| \leq 5$;
2. X is not a subset of any algebra in $\mathcal{V}_P$;
3. $\mathcal{V}$-SAT($\mathcal{S}$) is NP-hard if $\mathcal{V}$-SAT(X) is NP-hard since $X \subseteq \mathcal{S}$.

To show that $\mathcal{V}$-SAT($\mathcal{S}$) has to be NP-hard, a machine-assisted case analysis of the following form was performed:

1. Generate all subsets of $\mathcal{V}$ of size ≤ 5. There are
$$\sum_{i=0}^{5} \binom{32}{i} \approx 2.4 \times 10^5$$
such subsets.
2. Let $\mathcal{T}$ be such a set. Test: $\mathcal{T}$ is a subset of some subalgebra in $\mathcal{V}_P$ or $D \subseteq \mathcal{C}_{\mathcal{V}}(\mathcal{T})$ for some $D \in \mathcal{V}_{NP}$.

The test succeeds for all $\mathcal{T}$. Hence, $\mathcal{V}$-SAT($\mathcal{S}$) is NP-hard. Since $\mathcal{V}$-SAT is in NP [Meiri, 1996], NP-completeness follows. $\square$

	$\mathcal{V}^{23}$	$\mathcal{V}^{20}_{\mathtt{s}}$	$\mathcal{V}^{20}_{\mathtt{f}}$	$\mathcal{V}^{17}_{\mathtt{s}}$	$\mathcal{V}^{17}_{\mathtt{f}}$
$\perp$	•	•	•	•	•
(b)	•	•	•		
(s)	•	•		•	
(b s)	•	•	•	•	
(d)	•				
(b d)	•		•		
(s d)	•			•	
(b s d)	•		•	•	
(f)	•		•		•
(b f)			•		•
(s f)				•	•
(b s f)			•	•	•
(d f)	•				•
(b d f)	•		•		•
(s d f)	•			•	•
(b s d f)	•		•	•	•
(a)	•	•	•		
(b a)		•	•		
(s a)		•		•	
(b s a)		•	•	•	
(d a)	•	•			
(b d a)	•	•	•		
(s d a)	•	•		•	
(b s d a)	•	•	•	•	
(f a)	•	•	•		•
(b f a)		•	•		•
(s f a)		•		•	•
(b s f a)		•	•	•	•
(d f a)	•	•			•
(b d f a)	•	•	•		•
(s d f a)	•	•		•	•
$\top$	•	•	•	•	•

Table 3: The maximal subalgebras of $\mathcal{V}$ which have a polynomial-time satisfiability problem.

Subclass	Relations	Proof of NP-hardness
A_0	(d), (b a)	Lemma 2.6
A_1	(d), (b s a)	Lemma 2.7
A_2	(d), (b f a)	Lemma 2.7
A_3	(s d), (b a)	Lemma 2.7
A_4	(s d), (b f a)	Lemma 2.7
A_5	(d f), (b a)	Lemma 2.7
A_6	(d f), (b s a)	Lemma 2.7
A_7	(s d f), (b a)	Lemma 2.7
A_8	(d), (b s f a)	Lemma 2.7
B_0	(d), (b f)	Lemma 2.8
B_0^D	(d), (s a)	$B_0^D = \mathcal{D}_{\mathcal{V}}(B_0)$
B_1	(d), (b s f)	Lemma 2.9
B_1^D	(d), (s f a)	$B_1^D = \mathcal{D}_{\mathcal{V}}(B_1)$
B_2	(s d), (b f)	Lemma 2.9
B_2^D	(d f), (s a)	$B_2^D = \mathcal{D}_{\mathcal{V}}(B_2)$
B_3	(d a), (b f)	Lemma 2.10
B_3^D	(b d), (s a)	$B_3^D = \mathcal{D}_{\mathcal{V}}(B_3)$
B_4	(s d a), (b f)	Lemma 2.11
B_4^D	(b d f), (s a)	$B_4^D = \mathcal{D}_{\mathcal{V}}(B_4)$
B_5	(d a), (b s f)	Lemma 2.11
B_5^D	(b d), (s f a)	$B_5^D = \mathcal{D}_{\mathcal{V}}(B_5)$
B_6	(b f), (s a)	Lemma 2.12
C_0	(s), (b f)	Lemma 2.13
C_0^D	(f), (s a)	$C_0^D = \mathcal{D}_{\mathcal{V}}(C_0)$
D_0	(s f), (b d a)	Lemma 2.14
D_1	(s f), (b a)	Corollary 2.15
D_2	(s f), (b d)	Corollary 2.15
D_3	(s f), (d a)	Corollary 2.15
D_4	(s f), (d)	Corollary 2.15

Table 4: NP-hard subclasses of $\mathcal{V}$.

4 Conclusion

We have studied computational properties of the point-interval algebra. All of the 2^{32} possible subclasses are classified with respect to whether their corresponding satisfiability problem is in P or NP-complete. The classification reveals that there are exactly five maximal subclasses having a polynomial-time satisfiability problem.

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Appendix

This appendix collects the complexity results needed for the proof of Theorem 3.5. The proofs of membership in P can be found in part A and the NP-hardness results in part B.

1 Polynomial-Time Problems

Proving that $\mathcal{V}$-SAT($\mathcal{V}^{23}$) $\in$ P is straightforward.

Proposition 1.1 Deciding satisfiability of a set of PA formulae is a polynomial-time problem.

Proof: See [Vilain *et al.*, 1989]. □

Lemma 1.2 $\mathcal{V}$-SAT($\mathcal{V}^{23}$) is in P.

Proof: Follows immediately from the definition of $\mathcal{V}^{23}$ and the previous proposition. □

Before we can show that the other algebras in Table 3 have polynomial-time satisfiability problems, we need an auxiliary definition.

Definition 1.3 Let $S \subseteq \mathbb{R}$ be finite and denote the absolute value of x with $\mathrm{abs}(x)$. The *minimal distance in* S, $\mathrm{MD}(S)$, is defined as

$$\min\{\mathrm{abs}(x-y) \mid x, y \in S \text{ and } x \neq y\}.$$

Observe that $|S| \geq 2$ in order to make $\mathrm{MD}(S)$ defined. For all such S, $\mathrm{MD}(S) > 0$. The definition of minimal distance can be extended to $\mathcal{V}$-interpretations in the following way: Let $\Im$ be a $\mathcal{V}$-interpretation that assigns values to a set of points $\mathcal{P}$ and a set of intervals $\mathcal{I}$. Let

$$\mathrm{MD}(\Im) = \mathrm{MD}(\{\Im(p) \mid p \in \mathcal{P}\} \cup \{\Im(I^-), \Im(I^+) \mid I \in \mathcal{I}\}).$$

Lemma 1.4 $\mathcal{V}$-SAT($\mathcal{V}^{20}_{\mathsf{s}}$) is in P.

Proof: By Lemma 2.3, it is sufficient to show that $\mathcal{V}$-SAT($v_{\mathtt{s}}^{20}$) is in P. Let Θ be an arbitrary instance of $\mathcal{V}$-SAT($v_{\mathtt{s}}^{20}$). Define the function $\sigma : v_{\mathtt{s}}^{20} \rightarrow \{\leq, =, \neq, \top\}$ as follows: $\sigma((\mathtt{s}))$ = "$=$", $\sigma((\mathtt{b\ s}))$ = "$\leq$", $\sigma((\mathtt{b\ a}))$ = "$\neq$", $\sigma((\mathtt{b\ s\ d\ a}))$ = "$\top$" and $\sigma((\mathtt{b\ s\ f\ a}))$ = "$\top$".

Let $\Theta' = \{x_p\sigma(R)y_I | pRI \in \Theta\}$. In the sequel, y_I will be considered as the starting point of the interval I. By Proposition 1.1, it is a polynomial-time problem to decide PA-satisfiability of Θ'. We show that Θ is $\mathcal{V}$-satisfiable iff Θ' is PA-satisfiable.

if: Let $\Im'$ be a PA-model of Θ'. Let $\varepsilon = \mathrm{MD}(\Im')/2$. We construct an $\mathcal{V}$-interpretation $\Im$ of Θ as follows:

- For each variable x_p in Θ' let $\Im(p) = \Im'(x_p)$.
- For each variable y_I in Θ' let $\Im(I^-) = \Im'(y_I)$ and $\Im(I^+) = \Im'(y_I) + \varepsilon$

We show that $\Im$ is an $\mathcal{V}$-model of Θ. Let pRI be an arbitrary formula in Θ. We have five different cases.

1. $R = (\mathtt{s})$. Then $\Im'(x_p)\sigma((\mathtt{s}))\Im'(y_I) \Leftrightarrow \Im'(x_p) = \Im'(y_I) \Leftrightarrow \Im(p) = \Im(I^-)$. Consequently, $p(\mathtt{s})I$ under $\Im$.

2. $R = (\mathtt{b\ s})$. Then $\Im'(x_p)\sigma((\mathtt{b\ s}))\Im'(y_I) \Leftrightarrow \Im'(x_p) \leq \Im'(y_I) \Leftrightarrow \Im(p) \leq \Im(I^-)$. Consequently, $p(\mathtt{b\ s})I$ under $\Im$.

3. $R = (\mathtt{b\ a})$. Then $\Im'(x_p)\sigma((\mathtt{b\ a}))\Im'(y_I) \Leftrightarrow \Im'(x_p) \neq \Im'(y_I) \Leftrightarrow \Im(p) \neq \Im(I^-)$. If $\Im(p) < \Im(I^-)$ then $p(\mathtt{b})I$ under $\Im$. If $\Im(p) > \Im(I^-)$ then $\Im(p) > \Im(I^+)$ since $I^+ = I^- + \varepsilon$. In this case $p(\mathtt{a})I$ under $\Im$. Consequently, $p(\mathtt{b\ a})I$ under $\Im$.

4. $R = (\mathtt{b\ s\ f\ a})$. Assume $p(\mathtt{d})I$ under $\Im$. Then $\Im(I^-) < \Im(p) < \Im(I^+) \Leftrightarrow \Im(I^-) < \Im(p) < \Im(I^-) + \varepsilon \Leftrightarrow \Im'(y_I) < \Im'(x_p) < \Im'(y_I) + \varepsilon$ which contradicts the choice of ε. Hence, $p(\mathtt{b\ s\ f\ a})I$ under $\Im$.

5. $R = (\mathtt{b\ s\ d\ a})$. This case is analogous to the previous case.

only-if: Let $\Im$ be an $\mathcal{V}$-model of Θ. We construct a PA-interpretation $\Im'$ of Θ' as follows:

- For each point p in Θ let $\Im'(x_p) = \Im(p)$.
- For each interval I in Θ let $\Im'(y_I) = \Im(I^-)$.

Next, we show that $\Im'$ is a PA-model of Θ'. Let x_pRy_I be an arbitrary formula in Θ'. We have four different cases:

1. $R =$ "$=$". Assume to the contrary that $\Im'(x_p) \neq \Im'(y_I)$. Then $\Im(p) \neq \Im(I^-)$ and $p(\mathtt{b\ d\ f\ a})I$ under $\Im$. But $x_p = y_I \in \Theta'$ iff $p(\mathtt{s})I \in \Theta$ which leads to a contradiction. Hence, $x_p = y_I$ under $\Im'$.

2. $R =$ "$\neq$". This case is analogous to the previous case.

3. $R =$ "$\leq$". Assume $\Im'(x_p) > \Im'(y_I)$. Then $\Im(p) > \Im(I^-)$ and $p(\mathtt{d\ f\ a})I$ under $\Im$. But $x_p \leq y_I \in \Theta'$ iff $p(\mathtt{b\ s})I \in \Theta$ which leads to a contradiction. Hence, $x_p \leq y_I$ under $\Im'$.

4. $R =$ "$\top$". This relation holds trivially in every PA-model of Θ'. □

Corollary 1.5 $\mathcal{V}$-SAT$(\mathcal{V}_{\mathtt{f}}^{20})$ is in P.

Proof: $\mathcal{V}_{\mathtt{f}}^{20} = \mathcal{D}_{\mathcal{V}}(\mathcal{V}_{\mathtt{s}}^{20})$. □

Lemma 1.6 $\mathcal{V}$-SAT$(\mathcal{V}_{\mathtt{s}}^{17})$ is in P.

Proof: Let Θ be an arbitrary instance of $\mathcal{V}$-SAT$(\mathcal{V}_{\mathtt{s}}^{17})$. If a formula of the form $p \perp I$ is in Θ then Θ is not satisfiable. Otherwise, consider the following $\mathcal{V}$-interpretation: $\Im(p) = 0$ for every point p and $\Im(I^-) = 0$ and $\Im(I^+) = 1$ for every interval I. Let pRI be an arbitrary formula in Θ. By the definition of $\mathcal{V}_{\mathtt{s}}^{17}$, $\mathtt{s} \in R$. Obviously, $\Im$ satisfies pRI. Since it is a polynomial-time problem to check whether $p \perp I \in \Theta$ or not, the lemma follows. □

Corollary 1.7 $\mathcal{V}$-SAT$(\mathcal{V}_{\mathtt{f}}^{17})$ is in P.

Proof: $\mathcal{V}_{\mathtt{s}}^{17} = \mathcal{D}_{\mathcal{V}}(\mathcal{V}_{\mathtt{f}}^{17})$. □

2 NP-Hardness Results

This section provides NP-hardness proofs for the subclasses of $\mathcal{V}$ presented in Table 4. The reductions are mostly made from different subalgebras of Allen's interval algebra. Consequently, we begin this section by recapitulating some results concerning Allen's algebra. To make the proofs of NP-hardness less cumbersome, we will employ a technique which we refer to as *model transformations*; the definitions and results needed are collected in Section B.2.

Basic relation	Symbol	Example	Endpoint relations
X before Y	$\prec$	xxx	$X^- < Y^-$, $X^- < Y^+$,
Y after X	$\succ$	yyy	$X^+ < Y^-$, $X^+ < Y^+$
X meets Y	m	xxxx	$X^- < Y^-$, $X^- < Y^+$,
Y met-by X	$\mathsf{m}^{\smile}$	yyyy	$X^+ = Y^-$, $X^+ < Y^+$
X overlaps Y	o	xxxx	$X^- < Y^-$, $X^- < Y^+$,
Y overlapped-by X	$\mathsf{o}^{\smile}$	yyyy	$X^+ > Y^-$, $X^+ < Y^+$
X during Y	d	xxx	$X^- > Y^-$, $X^- < Y^+$,
Y includes X	$\mathsf{d}^{\smile}$	yyyyyyy	$X^+ > Y^-$, $X^+ < Y^+$
X starts Y	s	xxx	$X^- = Y^-$, $X^- < Y^+$,
Y started by X	$\mathsf{s}^{\smile}$	yyyyyyy	$X^+ > Y^-$, $X^+ < Y^+$
X finishes Y	f	xxx	$X^- > Y^-$, $X^- < Y^+$,
Y finished-by X	$\mathsf{f}^{\smile}$	yyyyyyy	$X^+ > Y^-$, $X^- < Y^+$
X equals Y	$\equiv$	xxxx yyyy	$X^- = Y^-$, $X^- < Y^+$, $X^+ > Y^-$, $X^+ = Y^+$

Table 5: The thirteen basic relations of the $\mathcal{A}$-algebra. The endpoint relations $X^- < X^+$ and $Y^- < Y^+$ that are valid for all relations have been omitted.

2.1 Allen's Algebra

Allen's interval algebra [Allen, 1983] is based on the notion of *relations between pairs of intervals*. An interval X is represented as an ordered pair $\langle X^-, X^+\rangle$ of real numbers with $X^- < X^+$, denoting the left and right endpoints of the interval, respectively, and relations between intervals are composed as disjunctions of *basic interval relations*. Their exact definitions can be found in Table 5. Such disjunctions are represented as sets of basic relations. The algebra is provided with the operations of *converse, intersection and composition* on intervals. The definitions of these operations can be found in [Allen, 1983]. By the fact that there are thirteen basic relations, we get $2^{13} = 8192$ possible relations between intervals in the full algebra. We denote the set of all interval relations by $\mathcal{A}$. The reasoning problem we will consider is the problem of *satisfiability* ($ISAT$) of a set of interval variables with relations between them, *i.e.* deciding whether there exists an assignment of intervals on the real line for the interval variables, such that all of the relations between the intervals hold. Such an assignment is said to be a $\mathcal{A}$-*model* for the interval variables and relations. For $\mathcal{A}$, we have the following result.

Theorem 2.1 Define the sets $\mathcal{N}_3$ and $\mathcal{F}$ as follows:

$$\mathcal{N}_3 = \{(\prec\ \succ), (\mathsf{o}\ \mathsf{o}^{\smile})\}$$

and

$$\mathcal{F} = \{(\equiv\ \mathsf{s}\ \mathsf{s}^{\smile}), (\prec\ \mathsf{d}^{\smile}\ \mathsf{o}\ \mathsf{m}\ \mathsf{m}^{\smile}\ \mathsf{f}^{\smile})\}.$$

$ISAT(\mathcal{N}_3)$ and $ISAT(\mathcal{F})$ are NP-hard problems.

Proof: The result for $\mathcal{N}_3$ can be found in [Drakengren and Jonsson, 1997b].

To prove that $ISAT(\mathcal{F})$ is NP-hard, we will use a closure operation for Allen's algebra which was defined by Nebel and Bürckert [Nebel and Bürckert, 1995]. Assume $\mathcal{S} \subseteq \mathcal{A}$. Then we denote by $\mathcal{C}_{\mathcal{A}}(\mathcal{S})$ the $\mathcal{A}$-*closure* of $\mathcal{S}$ under converse, intersection and composition, *i.e.* the least subset of $\mathcal{A}$ containing $\mathcal{S}$ closed under the three operations. Nebel and Bürckert [Nebel and Bürckert, 1995] have shown that $\mathcal{A}$-SAT($\mathcal{S}$) is NP-hard if and only if $\mathcal{A}$-SAT($\mathcal{C}_{\mathcal{A}}(\mathcal{S})$) is. Hence, to show NP-hardness of $\mathcal{A}$-SAT($\mathcal{F}$), we can study $\mathcal{A}$-SAT($\mathcal{C}_{\mathcal{A}}(\mathcal{F})$) instead of $\mathcal{A}$-SAT($\mathcal{F}$).

Now, consider the following set of relations:

$$\Delta_0 = \{(\equiv\ \mathsf{d}\ \mathsf{d}^{\smile}\ \mathsf{o}\ \mathsf{o}^{\smile}\ \mathsf{m}\ \mathsf{m}^{\smile}\ \mathsf{s}\ \mathsf{s}^{\smile}\ \mathsf{f}\ \mathsf{f}^{\smile}), (\prec\ \succ)\}.$$

Golumbic and Shamir [Golumbic and Shamir, 1993] have shown that $\mathcal{A}$-SAT(Δ_0) is NP-complete. It can be verified that Δ_0 is a subset of $\mathcal{C}_{\mathcal{A}}(\mathcal{F})$ and NP-hardness of $\mathcal{A}$-SAT($\mathcal{F}$) follows. □

In the previous lemma, $\mathcal{C}_{\mathcal{A}}$ was computed by the utility `aclose` [Nebel and Bürckert, 1993].

2.2 Model Transformations

One of our main vehicles for showing NP-hardness of different subclasses is that of *model transformations*. It is a method for transforming a solution of one problem to a solution of a related problem.

Definition 2.2 A *model transformation* is a mapping on $\mathcal{V}$-interpretations.

This definition is very general. To make it applicable in practice, we need a way of describing model transformations.

Definition 2.3 Let T be a model transformation. A function $f_T : \mathbf{B} \to 2^{\mathbf{B}}$ is a *description* of T iff for arbitrary $\mathcal{V}$-interpretations $\Im$, the following holds: if $b \in \mathbf{B}$ and pbJ under $\Im$ then $pf_T(b)J$ under $T(\Im)$. A description f_T can be extended to handle disjunctions in the obvious way: $f_T(R) = \bigcup_{r \in R} f_T(r)$.

Lemma 2.4 Let $\mathcal{R} = \{r_1, \ldots, r_n\} \subseteq \mathcal{V}$ and $\mathcal{R}' = \{r'_1, \ldots, r'_n\} \subseteq \mathcal{V}$ be such that $\mathcal{V}$-SAT$(\mathcal{R})$ is NP-hard and $r_k \subseteq r'_k$, $1 \leq k \leq n$. If there exists a model transformation T with a description f_T such that $f_T(r'_k) \subseteq r_k$ for every $1 \leq k \leq n$ then $\mathcal{V}$-SAT$(\mathcal{R}')$ is NP-hard.

Proof: Arbitrarily choose an instance Θ of $\mathcal{V}$-SAT$(\mathcal{R})$ and let $\Theta' = \{pr'I \,|\, prI \in \Theta\}$. Obviously, this is a polynomial-time transformation and Θ' is an instance of $\mathcal{V}$-SAT$(\mathcal{R}')$. We show that Θ is satisfiable iff Θ' is satisfiable.
only-if: Let $\Im$ be a model of Θ. Recall that $r_k \subseteq r'_k$, $1 \leq k \leq n$. Hence, $\Im$ is a model of Θ' since every formula $pr'I \in \Theta'$ is weaker than the corresponding formula $prI \in \Theta$.
if: Let $\Im'$ be a model of Θ'. We show that $T(\Im')$ is a model of Θ. Arbitrarily choose a formula prI in Θ. By the construction of Θ', there exists a formula $pr'I \in \Theta'$. Thus, we have $pr'I$ under $\Im'$ which implies $pf_T(r')I$ under $T(\Im')$ since f_T is a description of T. Furthermore, $f_T(r') \subseteq r$ so prI holds under $T(\Im')$. Since prI was arbitrarily chosen, $T(\Im')$ is a model of Θ. $\square$

Definition 2.5 Let $\Im$ be an arbitrary $\mathcal{V}$-interpretation and let $\varepsilon = \mathrm{MD}(\Im)/2$. Define the model transformation $T_{o,o'}$ for $o, o' \in \{+, -, 0\}$ as

1. for each point p let $T_{o,o'}(\Im)(p) = p$; and
2. for each interval I let $T_{o,o'}(\Im)(I^-) = I^- o \varepsilon$ if $o \in \{+, -\}$ and $T_{o,o'}(\Im)(I^-) = I^-$ otherwise; and
3. for each interval I let $T_{o,o'}(\Im)(I^+) = I^+ \; o' \; \varepsilon$ if $o' \in \{+, -\}$ and $T_{o,o'}(\Im)(I^+) = I^+$ otherwise.

Descriptions of the model transformations in the previous definition can be found in Table 6.

2.3 NP-Hard Subclasses of $\mathcal{V}$

Lemma 2.6 $\mathcal{V}$-SAT(A_0) is NP-hard.

	b	s	d	f	a
$T_{-,-}$	b	d	d	a	a
$T_{-,0}$	b	d	d	f	a
$T_{-,+}$	b	d	d	d	a
$T_{0,-}$	b	s	d	a	a
$T_{0,0}$	b	s	d	f	a
$T_{0,+}$	b	s	d	d	a
$T_{+,-}$	b	b	d	a	a
$T_{+,0}$	b	b	d	f	a
$T_{+,+}$	b	b	d	d	a

Table 6: Descriptions of some model transformations.

Proof: Reduction from $\mathcal{A}$-SAT($\mathcal{N}_3$) which is NP-hard by Lemma 2.1. Let I and J be two intervals. We will show how to express $I(\texttt{o}\ \texttt{o}^\smile)J$ and $I(\prec\ \succ)J$ in the point-interval algebra by only using the point-interval relations in A_0, *i.e.*, $(\texttt{d})$ and $(\texttt{b}\ \texttt{a})$. By doing so, we have shown NP-hardness of $\mathcal{V}$-SAT(A_0).

Assume we want to relate the intervals I and J with the relation $(\texttt{o}\ \texttt{o}^\smile)$. Observe that $I(\texttt{o}\ \texttt{o}^\smile)J$ holds iff there exist three real numbers a, b, c with the following properties:

$$I^- < a < I^+ \text{ but } a < J^- \text{ or } a > J^+$$

$$I^- < b < I^+ \text{ and } J^- < b < J^+$$

$$J^- < c < J^+ \text{ but } c < I^- \text{ or } c > I^+$$

Obviously, we can relate I and J with $(\texttt{o}\ \texttt{o}^\smile)$ by introducing three fresh points a, b, c and the following six point-interval formulae:

$$\begin{array}{ll} a(\texttt{d})I & a(\texttt{b}\ \texttt{a})J \\ b(\texttt{d})I & b(\texttt{d})J \\ c(\texttt{d})J & c(\texttt{b}\ \texttt{a})I. \end{array}$$

In order to relate two intervals I, J with the relation $(\prec\ \succ)$, we introduce two fresh intervals K, L and relate them with $(\texttt{o}\ \texttt{o}^\smile)$ by using three fresh points a, b and c as above. Observe that either $\Im(a) < \Im(b) < \Im(c)$ or $\Im(c) < \Im(b) < \Im(a)$ in any model $\Im$.

Now, relate a, b, c to I, J as follows:

$$\begin{array}{ll} a(\texttt{d})I & a(\texttt{b}\ \texttt{a})J \\ b(\texttt{b}\ \texttt{a})I & b(\texttt{b}\ \texttt{a})J \\ c(\texttt{b}\ \texttt{a})I & c(\texttt{d})J. \end{array}$$

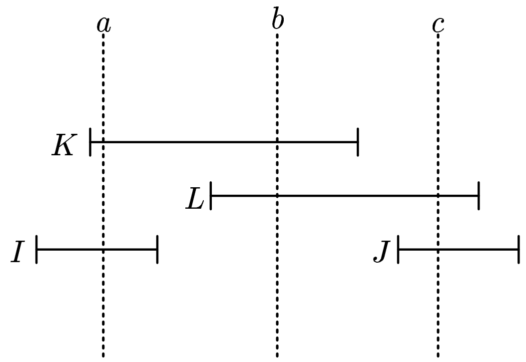

Figure 1: The construction of $I(\prec\ \succ)J$ in the proof of Lemma 2.6. The figure shows the situation when $\Im(I^+) < \Im(b) < \Im(J^-)$; the other case is analogous.

Let $\Im$ be a model satisfying these restrictions. It follows that either $\Im(I^+) < \Im(b) < \Im(J^-)$ or $\Im(J^+) < \Im(b) < \Im(I^-)$ and, consequently, $I(\prec\ \succ)J$. An explaination of this construction can be found in Figure 1. $\square$

Lemma 2.7 $\mathcal{V}$-SAT(A_i), $1 \leq i \leq 8$, is NP-hard.

Proof: Polynomial-time reduction from $\mathcal{V}$-SAT(A_0). Use the following model transformations

$$\begin{array}{ll} i=1: T_{+,0} & i=2: T_{0,-} \\ i=3: T_{-,0} & i=4: T_{-,-} \\ i=5: T_{0,+} & i=6: T_{+,+} \\ i=7: T_{-,+} & i=8: T_{+,-} \end{array}$$

and apply Lemma 2.4. $\square$

Lemma 2.8 $\mathcal{V}$-SAT(B_0) is NP-hard.

Proof sketch: The proof of Lemma 2.6 goes through if the relation (`b a`) is replaced by (`b f`). $\square$

Lemma 2.9 $\mathcal{V}$-SAT(B_i), $1 \leq i \leq 2$, is NP-hard.

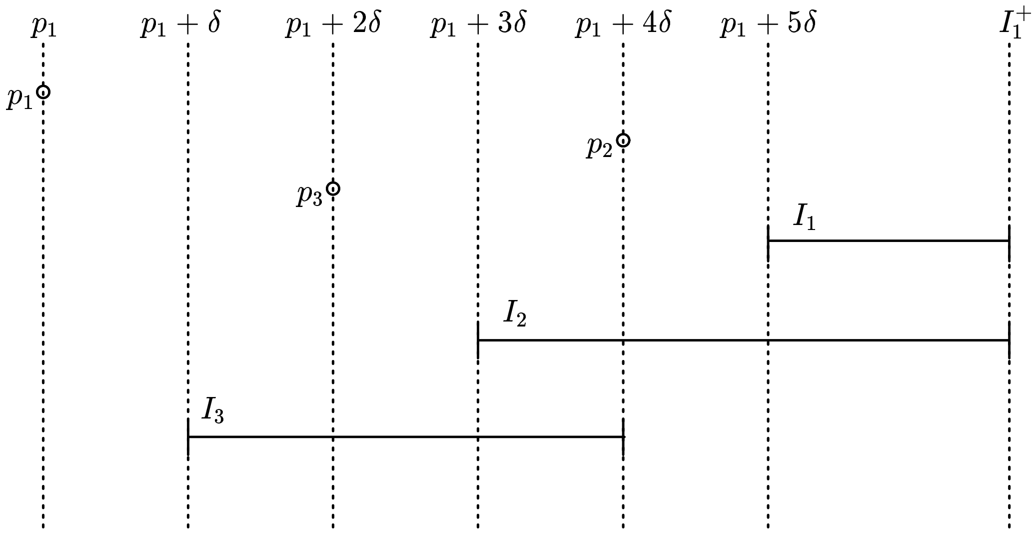

Figure 2: The construction used in Lemma 2.8.

Proof: Polynomial-time reduction from $\mathcal{V}$-SAT(B_0). Use the following model transformations

$$i = 1 : T_{+,0} \qquad i = 2 : T_{-,0}$$

and apply Lemma 2.4. $\square$

Lemma 2.10 $\mathcal{V}$-SAT(B_3) is NP-hard.

Proof: By Lemma 2.8, $\mathcal{V}$-SAT(B_0) is NP-hard. Let $E = \{(\mathtt{b}), (\mathtt{d\ a}), (\mathtt{b\ f})\}$. Then $(\mathtt{b\ f}) \otimes (\mathtt{d\ a}) \otimes (\mathtt{b}) = (\mathtt{b\ s\ d})$ and $(\mathtt{b\ s\ d}) \cap (\mathtt{d\ a}) = (\mathtt{d})$ so $B_0 \subseteq \mathcal{C}_{\mathcal{V}}(E)$ and $\mathcal{V}$-SAT(E) is NP-hard. Let Θ be an arbitrary instance of the $\mathcal{V}$-SAT(E) problem. We show how to construct an instance Θ' of the $\mathcal{V}$-SAT(B_3) problem that is satisfiable iff Θ is satisfiable.

We begin by showing how to relate a point p_1 and an interval I_1 such as $p_1(\mathtt{b})I_1$ by only using the relations in B_3. We introduce two fresh points p_2 and p_3 together with two fresh intervals I_2 and I_3. Consider the following construction:

$$\begin{array}{lll} (1)\ p_1(\mathtt{b\ f})I_1 & (2)\ p_1(\mathtt{b\ f})I_2 & (3)\ p_1(\mathtt{b\ f})I_3 \\ (4)\ p_2(\mathtt{b\ f})I_1 & (5)\ p_2(\mathtt{d\ a})I_2 & (6)\ p_2(\mathtt{b\ f})I_3 \\ (7)\ p_3(\mathtt{b\ f})I_1 & (8)\ p_3(\mathtt{b\ f})I_2 & (9)\ p_3(\mathtt{d\ a})I_3. \end{array}$$

We denote this set of formulae with Ω. Assume that $\Im$ is a $\mathcal{V}$-model of Ω. For the sake of brevity, we identify the points and intervals with their values when interpreted by $\Im$. Hence, instead of writing $\Im(p_1) < \Im(I_1^-)$, we simply write $p_1 < I_1^-$.

Obviously, $p_1 < I_1^-$ or $p_1 = I_1^+$. We begin by showing that there exists a $\mathcal{V}$-model of Ω such that $p_1 < I_1^-$. Let $\delta = (I_1^- - p_1)/5$. Consider the following assignment of values:

$$\begin{array}{lll} I_3^- = p_1 + \delta & p_3 = p_1 + 2\delta & I_2^- = p_1 + 3\delta \\ p_2 = p_1 + 4\delta & I_3^+ = p_1 + 4\delta & I_2^+ = I_1^+. \end{array}$$

This assignment is depicted in Figure 2. Obviously, the assignment is a $\mathcal{V}$-model of Ω.

Next, we show that there does not exist any $\mathcal{V}$-model of Ω such that $p_1 = I_1^+$. Assume $\Im$ is such a $\mathcal{V}$-model. By formula (4), we can see that $p_2 < I_1^-$ or $p_2 = I_1^+$. By assumption, $p_1 = I_1^+$. Hence, either $p_2 < I_1^-$ or $p_2 = p_1$. If $p_2 = p_1$ then formula (2) is equivalent to $p_2(\mathtt{b\ f})I_2$ which clearly contradicts formula (5). Thus, $p_2 < I_1^-$ and $p_2(\mathtt{b})I_1$. By analogous reasoning one can see that $p_3 < I_1^-$ and $p_3(\mathtt{b})I_1$.

Next, observe that formulae (2) and (3) imply $p_1 \le I_2^+$ and $p_1 \le I_3^+$. Furthermore, $p_2 < I_1^-$ and $p_3 < I_1^-$ which implies $p_2 < I_1^+$ and $p_3 < I_1^+$. By our initial assumption $p_1 = I_1^+$ we get

A: $p_2 < I_1^+ = p_1 \le I_3^+$

B: $p_3 < I_1^+ = p_1 \le I_2^+$

Consequently, $p_2 < I_3^+$ and $p_3 < I_2^+$. Observe that $p_2(\mathtt{b\ f})I_3$ and $p_3(\mathtt{b\ f})I_2$ by formulae (6) and (8), respectively. Hence, $p_2 < I_3^-$ and $p_3 < I_2^-$.

We have to study four cases, depending on how the $\mathcal{V}$-model assigns values to the variables in formula (5) and (9).

1. $I_2^- < p_2 < I_2^+$ and $I_3^- < p_3 < I_3^+$. Then $p_2 < I_3^- < p_3 < I_2^- < p_2$ which leads to a contradiction.

2. $I_2^+ < p_2$ and $I_3^+ < p_3$. Then $p_2 < I_3^- < I_3^+ < p_3 < I_2^- < I_2^+ < p_2$ which is a contradiction.

3. $I_2^+ < p_2$ and $I_3^- < p_3 < I_3^+$. Then $p_2 < I_3^- < p_3 < I_2^- < I_2^+ < p_2$. Contradiction.

4. $I_2^- < p_2 < I_2^+$ and $I_3^+ < p_3$. This case is analogous to the previous case.

Consequently, every $\mathcal{V}$-model of Ω satisfies $p_1 < I_1^-$.

We have thus shown how to express the relation $(\mathtt{b})$ by only using $(\mathtt{d\ a})$ and $(\mathtt{b\ f})$. Obviously, we can take an instance of the $\mathcal{V}$-SAT(E) problem and in polynomial time transform it into an equivalent instance of the $\mathcal{V}$-SAT(B_3) problem. NP-hardness of $\mathcal{V}$-SAT(B_3) follows immediately.

□

Lemma 2.11 $\mathcal{V}$-SAT(B_i), $4 \leq i \leq 5$, is NP-hard.

Proof: Polynomial-time reduction from $\mathcal{V}$-SAT(B_3). Use the following model transformations

$$i = 4 : T_{-,0} \qquad i = 5 : T_{+,0}$$

and apply Lemma 2.4. □

Lemma 2.12 $\mathcal{V}$-SAT(B_6) is NP-hard.

Proof: Let $E = \{(\mathtt{b}), (\mathtt{b\ f}), (\mathtt{s\ a})\}$. Then $(\mathtt{b\ f}) \otimes (\mathtt{s\ a}) \otimes (\mathtt{b}) = (\mathtt{b\ s\ d})$, $(\mathtt{s\ a}) \otimes (\mathtt{b}) \otimes (\mathtt{s\ a}) = (\mathtt{d\ f\ a})$ and $(\mathtt{b\ s\ d}) \cap (\mathtt{d\ f\ a}) = (\mathtt{d})$ so $B_0 \subseteq \mathcal{C}_{\mathcal{V}}(E)$ and $\mathcal{V}$-SAT(E) is NP-hard. Let Θ be an arbitrary instance of $\mathcal{V}$-SAT(E). We show how to construct an instance Θ' of $\mathcal{V}$-SAT(B_6) that is satisfiable iff Θ is satisfiable.

We begin showing how to relate a point p_1 and an interval I_1 such as $p_1(\mathtt{b})I_1$ by only using the relations in B_6. We introduce two fresh points p_2 and p_3 together with a fresh interval I_2. Consider the following construction:

$$\begin{array}{lll} (1)\ p_1(\mathtt{b\ f})I_1 & (2)\ p_1(\mathtt{b\ f})I_2 & (3)\ p_2(\mathtt{b\ f})I_1 \\ (4)\ p_2(\mathtt{s\ a})I_2 & (5)\ p_3(\mathtt{s\ a})I_1 & (6)\ p_3(\mathtt{b\ f})I_2 \end{array}$$

We denote this set of formulae with Ω. Let $\Im$ be a $\mathcal{V}$-model of Ω. As in the proof of Lemma 2.10, we identify the points and intervals with their value when interpreted by $\Im$.

Obviously, $p_1 < I_1^-$ or $p_1 = I_1^+$. We begin by showing that there exists a $\mathcal{V}$-model of Ω such that $p_1 < I_1^-$. Let $\Delta = (I_1^+ - I_1^-)/3$. Consider the following assignment of values:

$$p_3 = I_1^-$$

$$p_2 = I_1^+$$
$$I_2^- = I_1^+$$
$$I_2^+ = I_2^- + 1$$

It is not hard to see that this assignment is a $\mathcal{V}$-model of Ω.

Next, we show that there does not exist any $\mathcal{V}$-model of Ω such that $p_1 = I_1^+$. Assume $\Im$ is such a $\mathcal{V}$-model. We consider two cases:

Case 1: $p_1(\mathtt{f})I_2$. Then $p_1 = I_1^+ = I_2^+$. Hence, if $p_2(\mathtt{f})I_1$ then $p_2(\mathtt{f})I_2$ which contradicts formula (4). Consequently, $p_2(\mathtt{b})I_1$ and, by formula (4), $p_2(\mathtt{s})I_2$. Obviously, $I_2^- < I_1^- < I_2^+$.

By formula (5), $p_3(\mathtt{s\ a})I_1$. Assume $p_3(\mathtt{s})I_1$. Then $p_3(\mathtt{d})I_2$ which contradicts formula (6). Assume to the contrary that $p_3(\mathtt{a})I_1$. But this implies $p_3(\mathtt{a})I_2$ which also contradicts formula (6).

Case 2: $p_1(\mathtt{b})I_2$. Since $p_1(\mathtt{f})I_1$, I_1 and I_2 are disjoint intervals and I_1 is strictly before I_2. By formula (3), $p_2(\mathtt{b\ f})I_1$. If $p_2(\mathtt{b})I_1$ then $p_2(\mathtt{b})I_2$ which contradicts formula (4). If $p_2(\mathtt{f})I_1$ then $p_2(\mathtt{b})I_2$ which, again, contradicts formula (4).

We have thus shown how to express the relation $(\mathtt{b})$ by only using $(\mathtt{b\ f})$ and $(\mathtt{s\ a})$. Obviously, we can take an instance of the $\mathcal{V}$-SAT(E) problem and in polynomial time transform it into an equivalent instance of the $\mathcal{V}$-SAT(B_6) problem. NP-hardness of $\mathcal{V}$-SAT(B_6) follows immediately. □

Lemma 2.13 $\mathcal{V}$-SAT(C_0) is NP-hard.

Proof: Reduction from $\mathcal{A}$-SAT($\mathcal{F}$) which is NP-hard by Lemma 2.1. Let Θ be an instance of $\mathcal{A}$-SAT($\mathcal{F}$). We construct a set Θ' as follows.

1. For each formula of the type $I(\prec\ \mathtt{d}^\smile\ \mathtt{o\ m\ m}^\smile\ \mathtt{f}^\smile)J$ in Θ, introduce a new point $p_{I,J}$ and let $p_{I,J}(\mathtt{s})I$ and $p_{I,J}(\mathtt{b\ f})J$ in Θ';
2. For each formula of the type $I(\equiv\ \mathtt{s\ s}^\smile)J$ in Θ, introduce a new point $q_{I,J}$ and let $q_{I,J}(\mathtt{s})I$ and $q_{I,J}(\mathtt{s})J$ in Θ'.

Clearly Θ' is an instance of the $\mathcal{V}$-SAT(D_0) problem. It is a routine verification to show that Θ is satisfiable iff Θ' is satisfiable. □

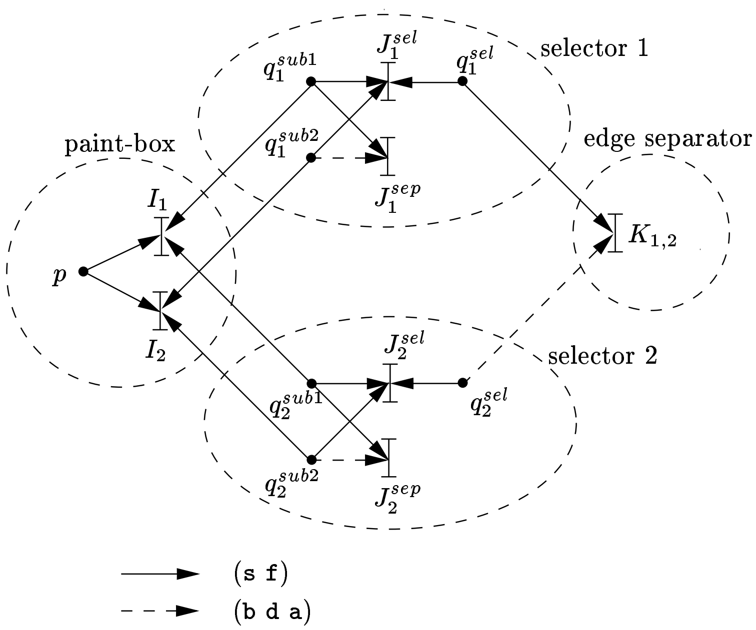

Figure 3: An example of the construction in the proof of Theorem 2.14 for a connected two-vertex graph.

Theorem 2.14 $\mathcal{V}$-SAT(D_0) is NP-hard.

Proof: Proof by reduction from GRAPH 3-COLOURABILITY, which is NP-complete [Garey and Johnson, 1979]. Let $G = \langle V, E \rangle$ be an undirected graph. Construct a corresponding set of point-interval formulae as follows (the construction is illustrated graphically in Figure 3 for the connected two-vertex graph $\langle \{v_1, v_2\}, \{\{v_1, v_2\}\} \rangle$).

First, we construct a *paint-box* defining our three "colours". It consists of a point p and two intervals I_1 and I_2, plus the two relations $p(\texttt{s f})I_1$ and $p(\texttt{s f})I_2$. Because of these constraints, the two intervals must have at least one end-point in common, so there is a total of two or three interval endpoints, which constitute our available colours. Hence, each model has a palette with two or three colours. We may, thus, occasionally get models corresponding to 2-colourings of G, but this is unimportant since every 2-colouring is a 3-colouring. Of course, the actual denotations of these colours differ between models, but this is also unimportant since they denote two or three different colours in each and every model.

Next, for each vertex $v_i \in V$ we introduce a *selector* consisting of two intervals J_i^{sel}, J_i^{sep} and three points q_i^{sub1}, q_i^{sub2} and q_i^{sel}. The interval J_i^{sep} acts as a *separator* for the points q_i^{sub1} and q_i^{sub2}. This means that by introducing the relations $q_i^{sub1}(\texttt{s f})J_i^{sep}$ and $q_i^{sub2}(\texttt{b d a})J_i^{sep}$ these two points are forced not to coincide. Further, both these points are related to the interval J_i^{sel} via the relations $q_i^{sub1}(\texttt{s f})J_i^{sel}$ and $q_i^{sub2}(\texttt{s f})J_i^{sel}$, which forces each endpoint of this interval to coincide with either of the two points. Then the point q_i^{sel} is related to interval J_i^{sel} as $q_i^{sel}(\texttt{s f})J_i^{sel}$, which forces it to coincide with either of the two points q_i^{sub1} and q_i^{sub2}. Finally, this whole gadget is connected to the paint-box via the relations $q_i^{sub1}(\texttt{s f})I_1$ and $q_i^{sub2}(\texttt{s f})I_2$. Hence, these two points select a subpalette of exactly two of the available colours, and the point q_i^{sel} then further selects one of these two colours as the colour of vertex v_i.

Finally, for each edge $\{v_i, v_j\} \in E$ we introduce an interval $K_{i,j}$ acting as a separator for the points q_i^{sel} and q_j^{sel}, in the same way as the separator interval within the selectors. More precisely, the relations $q_i^{sel}(\texttt{s f})K_{i,j}$ and $q_j^{sel}(\texttt{b d a})K_{i,j}$ are introduced. Obviously, these two formulae can be simultaneously satisfied iff vertices v_i and v_j have different colours, since the two points q_i^0 and q_j^0 must not coincide.

It is obvious that G is 3-colourable iff the set of point-interval formulae just constructed is $\mathcal{V}$-satisfiable, so NP-hardness of the algebra follows. $\square$

Corollary 2.15 $\mathcal{V}$-SAT(D_i), $1 \leq i \leq 4$, is NP-hard.

Proof: Use the same construction as in the proof of Theorem 2.14. We note that relations of the type $(\texttt{b d a})$ are used only in separators, forcing two points apart. Consider a separator interval I for two points p and q which are related as $p(\texttt{s f})I$ and $q\ r\ I$, where $r = (\texttt{b d a})$ in the original construction. We now consider how r can be constrained. First suppose that $p = q$. In this case at most one of the two relations must be satisfied, which holds as long as r does not contain either s or f. Conversely, suppose $p \neq q$. In this case, there must exist a choice of endpoints for I such that both formulae are satisfied. It is obviously a sufficient criterion that r contains either $\texttt{d}$ or both $\texttt{a}$ and $\texttt{b}$ (either of the latter alone is not sufficient, since other constraints in the model may dictate whether $p < q$ or $q < p$). It follows that r can be chosen as either of the relations $(\texttt{b a})$, $(\texttt{b d})$, $(\texttt{d a})$ and $(\texttt{d})$, which proves the corollary. $\square$

Paper II

Twenty-one Large Tractable Subclasses of Allen's Algebra

Thomas Drakengren and Peter Jonsson

ABSTRACT

This paper continues Nebel and Bürckert's investigation of Allen's interval algebra by presenting nine more maximal tractable subclasses of the algebra (provided that $P \neq NP$), in addition to their previously reported *ORD-Horn* subclass. Furthermore, twelve tractable subclasses are identified, whose maximality is not decided. Four of them can express the notion of *sequentiality* between intervals, which is not possible in the ORD-Horn algebra. All of the algebras are considerably larger than the ORD-Horn subclass. The satisfiability algorithm, which is common for all the algebras, is shown to be linear. Furthermore, the path consistency algorithm is shown to decide satisfiability of interval networks using any of the algebras.

1 Introduction

For specifying qualitative temporal information about relations between intervals, Allen's interval algebra [Allen, 1983] is often considered a convenient tool. However, due to its expressiveness (the satisfiability problem is NP-complete [Vilain *et al.*, 1989]), it is unlikely that there will be a polynomial-time algorithm for reasoning about the full algebra. Trying to overcome this, several tractable fragments of the algebra have been identified (e.g. [Nebel and Bürckert, 1995; van Beek, 1989; Golumbic and Shamir, 1993]), of which the largest known is Nebel and Bürckert's *ORD-Horn* algebra [Nebel and Bürckert, 1995]. Furthermore, this algebra has been proved to be the *unique maximal* algebra containing all the basic relations, comprising approximately 10 percent of the full algebra.

None of these algebras, however, is capable of expressing the notion of *sequentiality*, which is that of specifying that some intervals have to occur in sequence in time, without any overlap. This is required e.g. in some cases of reasoning about action [Sandewall, 1994]. The maximality result of the ORD-Horn algebra then implies that the requirement that an algebra should contain all the basic relations has to be sacrificed. Golumbic and Shamir [1993] come close to expressing sequentiality, but require that *any* two intervals are related, except for in an almost trivial four-element subset, which is not even an algebra. Four of our algebras strictly extend this subset.

In this paper, we exploit a simple graph algorithm, similar to that of van Beek [1992], and show that we can construct 21 large algebras for which this algorithm solves satisfiability in linear time, and furthermore, that four of these can express sequentiality, and nine of them are *maximal* tractable algebras (assuming $P \neq NP$, which we take for true in the rest of the paper). It should be noted that these algebras are of a size considerably larger than the ORD-Horn algebra: 20 contains 2178 elements, and one 4097 elements, which is half of Allen's algebra. However, this largest algebra has almost no expressiveness, showing that the size of an algebra need not have anything to do with its usefulness (as has often been argued in context with the ORD-Horn algebra).

The structure of the paper follows. In Section 2 and Section 3 we present the necessary background material about Allen's interval algebra, and some facts about the ORD-Horn algebra. Then, in Section 4, the concepts of "acyclic" and "DAG-satisfying" relations are introduced, after which the main results of the new tractable algebras are presented in Section 5. We show how the algorithm works in Section 6, and finally in Section 7, we show

Basic relation		Example	Endpoints
X before Y	$\prec$	xxx	$X^+ < Y^-$
Y after X	$\succ$	yyy	
X meets Y	m	xxxx	$X^+ = Y^-$
Y met-by X	$\mathsf{m}^\smile$	yyyy	
X overlaps Y	o	xxxx	$X^- < Y^- < X^+$,
Y overlapped-by X	$\mathsf{o}^\smile$	yyyy	$X^+ < Y^+$
X during Y	d	xxx	$X^- > Y^-$,
Y includes X	$\mathsf{d}^\smile$	yyyyyyy	$X^+ < Y^+$
X starts Y	s	xxx	$X^- = Y^-$,
Y started-by X	$\mathsf{s}^\smile$	yyyyyyy	$X^+ < Y^+$
X finishes Y	f	xxx	$X^+ = Y^+$,
Y finished-by X	$\mathsf{f}^\smile$	yyyyyyy	$X^- > Y^-$
X equals Y	$\equiv$	xxxx	$X^- = Y^-$,
		yyyy	$X^+ = Y^+$

Table 1: The thirteen basic relations. The endpoint relations $X^- < X^+$ and $Y^- < Y^+$ that are valid for all relations have been omitted.

that the *path consistency algorithm* is sufficient for deciding consistency for any of these algebras. A discussion concludes the paper.

This paper is an extended and modified version of an earlier paper [Drakengren and Jonsson, 1996]. The main additions are Section 6 and Section 7, but we have also improved the presentation of the algebras considerably.

2 Allen's Interval Algebra

Allen's interval algebra [Allen, 1983] is based on the notion of *relations between pairs of intervals*. An interval X is represented as an ordered pair $\langle X^-, X^+\rangle$ of real numbers with $X^- < X^+$, denoting the left and right endpoints of the interval, respectively, and relations between intervals are composed as disjunctions of *basic interval relations*, which are those in Table 1. Such disjunctions are represented as *sets* of basic relations, but using a notation such that, for example, the disjunction of the basic interval relations $\prec$, m and $\mathsf{f}^\smile$ is written $(\prec\ \mathsf{m}\ \mathsf{f}^\smile)$. Thus, we have that $(\prec\ \mathsf{f}^\smile) \subseteq (\prec\ \mathsf{m}\ \mathsf{f}^\smile)$.

The algebra is provided with the operations of *converse, intersection and* *composition* on interval relations:

- The converse operation takes an interval relation i to its converse $i^{\smile}$, obtained by inverting each basic relation in i, by exchanging X and Y in the endpoint relations of Table 1.
- The intersection operation takes two interval relations R_1 and R_2 to their intersection $R_1 \cap R_2$, by taking the basic relations that are contained in both R_1 and R_2.
- The composition operation takes two interval relations R_1 and R_2 to their composition $R_1 \circ R_2$, which is the Allen relation R_3 such that IR_3J iff $\exists K. IR_1K \wedge KR_2J$.

By the fact that there are thirteen basic relations, we get $2^{13} = 8192$ possible relations between intervals in the full algebra. We denote the set of all interval relations by $\mathcal{A}$. Subclasses of the full algebra are obtained by considering subsets of $\mathcal{A}$.

Although there are several computational problems associated with Allen's interval algebra, this paper focuses on the problem of *satisfiability* of a set of interval variables with relations between them, i.e. deciding whether there exists an assignment of intervals on the real line for the interval variables, such that all of the relations between the intervals hold. We define this as follows.

Definition 2.1 ($ISAT(S)$) Let $S \subseteq \mathcal{A}$ be a set of interval relations. An instance of $ISAT(S)$ is a labelled directed graph $G = \langle V, E\rangle$, where the nodes in V are interval variables and E is a subset of $V \times S \times V$. A labelled edge $\langle u, r, v\rangle \in E$ means that u and v are related by the relation r.

A function M taking an interval variable v to its interval representation $M(v) = \langle x^-, x^+\rangle$ with $x^- < x^+$, is said to be an *interpretation* of G.

An instance $G = \langle V, E\rangle$ is said to be *satisfiable* iff there exists an interpretation M such that for each $\langle u, r, v\rangle \in E$, $M(u)rM(v)$ holds, i.e., the endpoint relations required by r (see Table 1) are satisfied by the assignments of u and v. Then M is said to be a *model* of G.

We refer to the *size* of an instance G as $|V| + |E|$, as usual. $\square$

For $\mathcal{A}$, we have the following result.

Proposition 2.2 $ISAT(\mathcal{A})$ is NP-complete.
Proof: See Vilain et al [1989]. $\square$

The following auxiliary concept shall be needed.

Definition 2.3 ("satisfied as") Let $\mathcal{I} = \langle V, E \rangle$ be an instance of the satisfiability problem, M a model for $\mathcal{I}$, $\langle I_1, r, I_2 \rangle \in E$, and $r' \subseteq r$. Then r is said to be *satisfied as r' in M* iff $I_1 r' I_2$ is satisfied in M. $\Box$

Example 2.4 Let I_1, I_2 be interval variables related by $I_1(\prec \succ)I_2$, and M a model where I_1 is interpreted as $[1,2]$ and I_2 as $[3,4]$. Then in M, $(\prec \succ)$ is satisfied as $(\prec)$, but also as $(\prec \succ)$. $\Box$

3 The ORD-Horn Subclass

Nebel and Bürckert [1995] identify a subclass of the interval algebra, having the property that it is a maximal subclass containing all the basic interval relations, for which satisfiability can be solved using a polynomial-time algorithm, and is in fact the *unique* such maximal class[1]. This algebra, the *ORD-Horn* algebra, contains 868 relations, and thus covers slightly more than 10 percent of $\mathcal{A}$.

One of the main tools for analysing the ORD-Horn subclass is a *closure operation* on subclasses of the algebra, which preserves tractability.

Definition 3.1 (Closure) Let $S \subseteq \mathcal{A}$. Then we denote by $\overline{S}$ the *closure* of S under converse, intersection and composition, i.e. the least subalgebra containing S closed under the three operations. $\Box$

The key for extrapolating intractability results is the following.

Proposition 3.2 Let $S \subseteq \mathcal{A}$. Then $ISAT(S)$ is polynomial iff $ISAT(\overline{S})$ is, and $ISAT(S)$ is NP-complete iff $ISAT(\overline{S})$ is.
Proof: See Nebel and Bürckert [1995]. $\Box$

4 Acyclic and DAG-satisfying Relations

This section introduces some auxiliary notions and results needed for defining the new algebras, and proving their properties. In the rest of this paper, we let G denote an instance of $ISAT(X)$ instance for some $X \subseteq \mathcal{A}$.

[1]The uniqueness is proved under the assumption that the subclass must contain the empty relation ( ) and the full relation ($\equiv \prec \succ$ d d$^{\smile}$ o o$^{\smile}$ m m$^{\smile}$ s s$^{\smile}$ f f$^{\smile}$).

Definition 4.1 (Acyclic relation) A relation r is said to be an *acyclic relation* iff for every G and every cycle C in G such that every arc in C is labelled with r, C is never satisfiable. $\square$

Example 4.2 $\prec$ is an acyclic relation, and so is $(\prec\ \mathsf{m})$. $\square$

Corollary 4.3 Let r be an acyclic relation. Then every relation $r' \subseteq r$ is acyclic.
Proof: Since taking subsets of r constrains satisfiability further, the result follows. $\square$

Corollary 4.4 Let r be an acyclic relation, and A such that $A \subseteq \{r' | r' \subseteq r\}$. Then, for every G and every cycle C in G such that every arc in C is labelled by some relation in A, C is unsatisfiable.
Proof: Same argument as in Corollary 4.3. $\square$

Definition 4.5 (Maximal acyclic relation) An acyclic relation r for which there is no acyclic relation $r' \supset r$, is said to be a *maximal acyclic relation.* $\square$

In Proposition 4.8, we shall list all possible maximal acyclic relations.

Proposition 4.6 Let r be an acyclic relation, and A, A' sets such that $A \subseteq \{r' | r' \subseteq r\}$, and $A' = \{a \cup (\equiv) | a \in A\} \cup \{(\equiv)\}$. Then, every cycle C labelled by relations in $A \cup A'$ is satisfiable iff it contains only relations from A', and, furthermore, in every model of C all relations in the cycle have to be satisfied as $\equiv$.
Proof:
$\Rightarrow$) Suppose that a cycle C is satisfiable, and that it contains some relation from A. Induction on the number n of arcs in the cycle. For $n = 1$, we get a contradiction by the assumption. So, suppose for the induction that C contains $n + 1$ arcs. Let M be a model for the relations in C. It cannot hold that every relation in C is satisfied in M as some relation in A, by Corollary 4.4. Thus, some relation r' in C has to be satisfied as $\equiv$. But then we can collapse the two interval variables connected by r' to one interval variable, and we have a cycle of size n containing a relation from A. This contradicts the induction hyphothesis.
$\Leftarrow$) Suppose that a cycle C contains only relations in A'. Then C can be satisfied by choosing $\equiv$ on every arc, thus forcing the satisfying intervals to be identical. $\square$

An example is in order.

Example 4.7 Let $r = (\prec \quad \mathsf{m})$ (which is easily verified to be acyclic), $A = \{(\prec \quad \mathsf{m}), (\prec), (\ )\}$, and thus $A' = \{(\equiv\ \prec \quad \mathsf{m}), (\equiv\ \prec), (\equiv)\}$. Now take a cycle $I_1(\prec\ \mathsf{m})I_2(\equiv\ \prec)I_3(\equiv\ \prec\ \mathsf{m})I_1$. It is clear that this cycle is unsatisfiable, since it contains one relation which is not in A'. The cycle $I_1(\equiv\ \prec\ \mathsf{m})I_2(\equiv)I_3(\equiv\ \prec)I_1$ is clearly satisfiable taking $\equiv$ on each arc, but is not satisfiable in any other way, by the result. □

Next, we find all possible acyclic relations. The second part of the result, that there are no other maximal acyclic relations, is not used in any of the later results of the paper. However, it is important to know that there is no need to search for more maximal acyclic relations in future research.

Proposition 4.8 The only maximal acyclic relations in $\mathcal{A}$ are

$$\begin{array}{ll} (\prec\ \mathsf{d}^{\smile}\ \mathsf{o}\ \mathsf{m}\ \mathsf{s}\ \mathsf{f}^{\smile}), & (\prec\ \mathsf{d}^{\smile}\ \mathsf{o}\ \mathsf{m}\ \mathsf{s}^{\smile}\ \mathsf{f}^{\smile}), \\ (\prec\ \mathsf{d}\ \mathsf{o}\ \mathsf{m}\ \mathsf{s}\ \mathsf{f}), & (\prec\ \mathsf{d}\ \mathsf{o}\ \mathsf{m}\ \mathsf{s}\ \mathsf{f}^{\smile}), \end{array}$$

and their respective converses.

Proof: Obviously, a maximal acyclic relation cannot contain both a basic relation and its converse, and thus cannot contain $\equiv$. One consequence of this is that a maximal acyclic relation cannot contain more than six basic relations. So, if the above relations are shown to be acyclic, then they are also maximal.

Now, consider Table 2, which extracts from Table 1 how the basic relations (except for $\equiv$) relate the ending points of intervals. The table is to be read as follows. Suppose that the intervals i_1 and i_2 are related by some basic relation b, i.e. $i_1(b)i_2$, and consider the l row entry for b.

- If it is $+$, then the starting point of i_2 must be strictly *after* the starting point of i_1
- If it is $-$ then the starting point of i_2 must be strictly *before* the starting point of i_1
- If it is $=$, then the starting points of i_1 and i_2 have to coincide.

Similarly, the r row states the same information for the ending points.

Now consider the l row. If we choose a relation r' to contain exactly the basic relations which have a $+$ there, we know that r' will be an acyclic relation, because if in a cycle, the left ending points of the intervals have to increase at every arc, it cannot be satisfied. In addition to those basic relations in r', we can include in r' one basic relation b' which has an $=$ in the l row, yielding the relation r'', since then, a cycle labelled by r'' on every arc has to be satisfied as b' on every arc (otherwise, we would get a contradiction, by strictly increasing starting point values). But since neither of s and $\mathsf{s}^{\smile}$ has an $=$ in their r row, this is impossible. This gives us two choices of acyclic relations, which are the two first ones listed.

Symmetrically, by inspecting the r row, we see that we get the next two relations listed. Finally, by taking the $-$ entries instead of the $+$ entries, we get the converse relations of the listed ones.

It remains to prove that these are the only maximal acyclic relations. So, suppose that some acyclic relation e is not a subset of (or equal to) any of the relations in the statement of the proposition. First, note that e cannot be a basic relation, since every basic acyclic relation is included in some of the listed relations. Thus, e has to contain at least two distinct basic relations b_1 and b_2. Without loss of generality (using Corollary 4.3), we have that $e = (b_1\ b_2)$.

By the choice of the listed relations, b_1 and b_2 must have opposite signs either in their l or r rows (or both). Suppose that b_1 and b_2 do not have opposite signs in their l row, i.e. that either they have the same sign, or at least one of them has an $=$. If both of them have an $=$, they have to be s and $\mathsf{s}^{\smile}$, which is impossible. If they have the same sign, which is not $=$, then they are included in one of the listed relations, by definition. If at least one of them, say b_1, has $=$ there, i.e. b_1 is either s or $\mathsf{s}^{\smile}$, we see that for any basic acyclic relation c, c and b_1 occur together in some of the listed relations (or their converses), and in particular, this holds when c is b_2. Thus b_1 and b_2 have to have opposite signs in the l row. Symmetrically, b_1 and b_2 must have opposite signs also in the r row.

Now, the only remaining choice of b_1 and b_2, for which the signs of the l and r rows do not coincide, is for the basic intervals d and $\mathsf{d}^{\smile}$. But trivially, these cannot together be part of any acyclic relation, and thus b_1 and b_2 have to be chosen such without loss of generality, b_1 has $+$ in both its l and r rows, and similarly for b_2, $-$ in both its l and r rows. Obviously, also every choice when b_1 and b_2 are converses is impossible.

This leaves us with six relations to check: $(\succ\ \mathsf{m})$, $(\mathsf{o}^{\smile}\ \mathsf{m})$ and $(\prec\ \mathsf{o}^{\smile})$ and their converses, and it is enough to check the first three due to symmetry. Now, it is easy to construct satisfiable cycles using relations containing either of these relations. $\Box$

	m	m$^\smile$	$\prec$	$\succ$	o	o$^\smile$
l	$+$	$-$	$+$	$-$	$+$	$-$
r	$+$	$-$	$+$	$-$	$+$	$-$

	d	d$^\smile$	f	f$^\smile$	s	s$^\smile$
l	$-$	$+$	$-$	$+$	$=$	$=$
r	$+$	$-$	$=$	$=$	$+$	$-$

Table 2: The effect of relations on interval endpoints.

Definition 4.9 (DAG-satisfying relation) A basic relation b is said to be *DAG-satisfying* iff any DAG (directed acyclic graph) labelled only by relations containing b is satisfiable. $\Box$

Now, we shall classify the DAG-satisfying relations, after an auxiliary definition.

Definition 4.10 (Source) Let G be a DAG. Then a node v in G is said to be a *source* iff there are no arcs which end in v. $\Box$

Proposition 4.11 The basic relations $\prec$, d, o, f, s and $\equiv$, and their respective converses, are DAG-satisfying.

Proof: We show that any DAG labelled only by relations containing a fixed basic relation b, when b is one of the above relations, is satisfiable with some model M. Indeed, we prove the stronger result[2] that we can choose the satisfying M such that

- when b is d or o, all intervals overlap at some open interval
- when b is f, every interval has the same right ending point
- when b is s, every interval has the same left ending point
- when b is $\equiv$, all intervals are identical.

The result for the converse relations follows by an analogous construction.

Induction on the number of nodes in the DAG G. The case when $n = 0$ is trivial. Suppose that G has $n+1$ elements, and remove a source g from G.

[2]When b is $\prec$, we need not strengthen the result.

By induction, the remaining graph G' is satisfiable by a model M satisfying the required condition for the relation b. We shall now construct a model M' of G, which agrees with M on every interval variable in G'. The satisfying interval, denoted s, for the remaining interval variable represented by the node g, is chosen as follows, depending on b and M. Note that M satisfies the above conditions.

- When b is $\prec$, choose s to be any interval strictly before every interval in M

- When b is d, choose s to be an interval which is within the common open interval of the intervals in M

- When b is o, choose s to have its left ending point to the left of every interval in M, and its right ending point to be in the middle of the common interval of the intervals in M

- When b is f, choose s to have the same right ending point as the intervals in M, and the left ending point to be in the middle of the interval in M which has the rightmost left ending point

- When b is s, choose s to have the same left ending point as the intervals in M, and the right ending point to be in the middle of the interval in M which has the leftmost right ending point

- When b is $\equiv$, choose s to be identical to the intervals in M.

Obviously, M' is a model of G satisfying the requirements. □

We may note that m is not DAG-satisfying: take interval variables I_1, I_2 and I_3 related by $I_1(\mathsf{m})I_2$, $I_2(\mathsf{m})I_3$ and $I_1(\mathsf{m})I_3$. This is a DAG which is not satisfiable.

5 Tractable Algebras

Now we define the algebras which are to be analysed[3].

[3]This definition differs slightly from that of [Drakengren and Jonsson, 1996], but is easily verified (using Nebel and Bürckert's software [1993]) to result in the same maximal tractable algebras.

Definition 5.1 (The subclasses $A(r,b)$) Let b be a DAG-satisfying basic relation and r an acyclic relation containing b. First define the subclasses $A_1(b)$, $A_2(r,b)$ and $A_3(r,b)$ by

$$A_1(b) = \{r' \cup (b\ b^{\smile}) | r' \in \mathcal{A}\},$$

$$A_2(r,b) = \{r' \cup (b) | r' \subseteq r\}$$

and

$$A_3(r,b) = \{r' \cup (\equiv) | r' \in A_2(r,b)\} \cup \{(\equiv)\}.$$

Then set

$$B = A_1(b) \cup A_2(r,b) \cup A_3(r,b)$$

and finally define the subclass $A(r,b)$ by

$$A(r,b) = B \cup \{x^{\smile} | x \in B\} \cup \{(\ )\}.$$

□

Corollary 5.2 Let r be an acyclic relation, $r' \subseteq r$, and b be some DAG-satisfying basic relation. Then $A(r',b) \subseteq A(r,b)$.
Proof: By the construction of $A(r,b)$. □

Thus, by Corollary 5.2 and Corollary 4.3, it is sufficient to use maximal acyclic relations when constructing the algebras $A(r,b)$.

Now, using Proposition 4.8 we can construct twenty $A(r,b)$, by choosing r to be one of the maximal acyclic relations above, and choosing b to be an element in the chosen r except for m or $\mathsf{m}^{\smile}$. The reason why we get only twenty combinations (and not forty) is that the algebras are closed under the converse operation. Note that this exhausts the choices of parameters in $A(\cdot,\cdot)$, by Corollary 5.2 and Proposition 4.8.

We now prepare for an explicit listing of the algebras.

Proposition 5.3 Let r be maximal acyclic, $b \in r$ and $b \notin \{\mathsf{m}, \mathsf{m}^{\smile}\}$. Then $A(r,b)$ is the set of all $r' \in \mathcal{A}$ satisfying one of the following inclusions:

$$\begin{array}{lllll} (b\ b^{\smile}) & \subseteq & r' & & \\ (b) & \subseteq & r' & \subseteq & (\equiv) \cup r \\ (b^{\smile}) & \subseteq & r' & \subseteq & (\equiv) \cup r^{\smile} \\ & & r' & \subseteq & (\equiv). \end{array}$$

Proof: An easy comparison with the definitions. □

Using this, we can list the algebras in a reasonably compact way. See the appendix for a complete listing of them all.

Proposition 5.4 Each of the $A(r,b)$ sets are algebras containing 2178 elements, and each contains exacly three basic relations, namely $\equiv$, b and $b^{\smile}$. Furthermore, all of these twenty algebras are distinct.
Proof: That the sets are algebras (i.e. closed under converse, intersection and composition) is verified by running the utility `aclose` by Nebel and Bürckert [1993]. The sizes of the algebras, distinctness, and what basic relations are included, can easily be obtained from the explicit listings above. □

We have four algebras $A(r,\prec)$, all containing the relations $(\equiv)$, $(\prec)$, $(\prec\ \equiv)$, $(\succ)$, $(\succ\ \equiv)$ and $(\prec\ \succ)$, expressing the notion of *sequentiality*, which is useful for solving reasoning problems under the assumption that actions always occur in sequence [Sandewall, 1994]. Note that the ORD-Horn algebra does *not* contain the relation $(\prec\ \succ)$, and thus cannot express sequentiality.

We now state the algorithm which we shall show in Theorem 5.9 solves satisfiability for these algebras, after a short definition.

Definition 5.5 (Strong component) A subgraph C of a graph G is said to be a *strong component* of G iff it is maximal such that for any nodes a, b in C, there is always a path in G from a to b. □

In fact, this algorithm is very similar to that of van Beek [1992], improved and used by Gerevini *et al.* [1993], but here used on intervals instead of points. In Section 6, we will show how the algorithm runs on an example.

We now state a simple result which holds for directed graphs in general.

Proposition 5.7 Let G be loop-free[4] with an acyclic subgraph D. Then those arcs of G which are not in D can be reoriented so that the resulting graph is acyclic.
Proof: Induction over the number n of nodes in G that are not in D. For $n = 0$, the result is trivial. So, suppose that there are $n+1$ nodes in G that are not in D, and remove an arbitrary node v of these, resulting in the graph G'. By induction, the arcs of G' can be reoriented to form a DAG G''. Now

[4] A graph is said to be loop-free if it has no arcs from a node v to the node v.

Algorithm 5.6 ($ISAT(A(r,b))$)

```
1  Redirect the arcs of G so that all relations
   are in A1(b) ∪ A2(r,b) ∪ A3(r,b)
2  Let G' be the graph obtained from G by removing arcs which are not
   labelled by some relation in A2(r,b) ∪ A3(r,b)
3  Find all strong components C in G'
4  for every arc e in G whose relation
      does not contain ≡
5     if e connects two nodes in some C then
6        reject
7     endif
8  endfor
9  accept
```

□

add the node v to G'' obtaining G''', and reorient any arcs between G'' and v (in either direction) towards v. Since the graph is loop-free, no cycles are added by this operation, so G''' is acyclic. □

We now specialise this result.

Corollary 5.8 Let G be loop-free with an acyclic subgraph D, b a DAG-satisfying basic relation, and let the arcs of D be labelled by relations containing b, and the arcs not in D be labelled by relations containing both b and $b^{\smile}$. Then G is satisfiable.

Proof: Reorient the arcs of G like in Proposition 5.7, yielding a DAG G'. In this construction, whenever an arc is reoriented, also invert the relation on that arc, so that G' is satisfiable iff G is. By the construction, only arcs containing *both* b and $b^{\smile}$ have been reoriented, so every arc in the DAG G' contains b and, thus, since b is DAG-satisfying, G' is satisfiable, and consequently, also G is satisfiable. □

Theorem 5.9 Algorithm 5.6 correctly solves satisfiability for $A(r,b)$.

Proof: First, note that line 1 does not change satisfiability of G and can always be done according to the definition of $A(r,b)$.

Suppose that the algorithm rejects. Then there is an arc e not containing $\equiv$ that connects two nodes within a strong component C in G'. Suppose that $e \in A_2(r,b)$. Then there is a cycle of relations in $A_2(r,b) \cup A_3(r,b)$ containing e, and by Proposition 4.6, since e does not contain $\equiv$, C is unsatisfiable. Thus, suppose that $e \notin A_2(r,b)$ and that C does not contain any relation from $A_2(r,b)$. Then by Proposition 4.6 all relations in C have to be satisfied as $\equiv$. But since e does not contain $\equiv$, unsatisfiability results.

Now suppose that the algorithm accepts. Thus every strong component C can be collapsed to one interval, removing all arcs which would start and end in the collapsed interval, retaining the same condition for satisfiability, using the same argument as above. After the collapsing, the subgraph obtained by considering only arcs labelled by relations in $A_2(r,b) \cup A_3(r,b)$ will be acyclic. Since by construction every relation in $A_2(r,b) \cup A_3(r,b)$ contains the relation b, and the remaining arcs are labelled by relations containing both b and $b^{\smile}$, the graph is satisfiable by Corollary 5.8 (note that the graph will be loop-free, since every node is contained in some strong component). $\Box$

Theorem 5.10 Algorithm 5.6 runs in linear time in the size of G.
Proof: Line 1 is easily done in linear time. Strong components can be found in linear time (see e.g. Baase [1988]). Also the final test can be done in linear time. $\Box$

Corollary 5.11 Satisfiability of $A(r,b)$ is solvable in linear time.
Proof: From Theorem 5.9 and Theorem 5.10. $\Box$

Next, for the maximality results.

Proposition 5.12 The eight algebras $A(r,b)$ which have $b \in \{\mathsf{f}, \mathsf{s}\}$ are maximal tractable algebras.
Proof: By running the utility `atry` [Nebel and Bürckert, 1993], which generates minimal extensions of subclasses by adding a relation and computing the closure of that class. For these algebras, no nontrivial extensions were found (i.e. every extension results in $\mathcal{A}$), and since $ISAT(\mathcal{A})$ is NP-complete by Proposition 2.2, the result follows by Proposition 3.2. $\Box$

The remaining algebras are not maximal, as shown by Drakengren and Jonsson [1997a], where each is extended with 134 elements with retained tractability. However, the present algorithm has a lower asymptotic complexity than the one that is needed for the extended algebras.

Finally, we cover a case which is related to the $A(\cdot,\cdot)$ algebras, but occurs when every relation contains $\equiv$.

Definition 5.13 (The algebra $A_{\equiv}$) Define the algebra $A_{\equiv}$ to contain every relation that contains $\equiv$, and the empty relation $(\ )$. It is easy to see that $A_{\equiv}$ contains 4097 elements. □

For this algebra, we have the following trivial algorithm.

Algorithm 5.14 (Satisfiability in $A_{\equiv}$)

```
if some arc is labelled by ( ) then
   reject
else
   accept
endif
```

□

Proposition 5.15 Algorithm 5.14 correctly solves satisfiability in $A_{\equiv}$ in linear time. Furthermore, it is a maximal tractable subclass of $\mathcal{A}$.
Proof: Correctness and complexity results are trivial. The maximality follows by running the utility `atry` [Nebel and Bürckert, 1993], which generates no nontrivial extensions of the algebra. □

The algebra $A_{\equiv}$ certainly raises doubts about whether the *size* of a subalgebra can be used to judge its usefulness, since its expressivity is obviously too weak to be of any use.

6 Example

In this section we show how to model an industrially-inspired problem using one of the algebras, and exemplify how the algorithm runs on this problem.

Consider the following fragment of a chemical process. First denote different chemical substances by letters $u, v, \ldots, z$ possibly with primes on them. Then we have seven subprocesses $P_1, \ldots, P_7$, operating as follows.

P_1 melts x, producing x',
P_2 melts y and adds u', producing y',
P_3 melts z, producing z',
P_4 heats y', producing y'',
P_5 mixes x', y'', z' and w', producing x'',
P_6 pulverises u, producing u', and
P_7 cools w, and u' is added, producing w'.

Initially, all "unprimed" chemicals are available, whereas the "primed" ones are to be produced. The process itself, together with its controllers, impose the following ordering restrictions. The processes $P_1, \ldots, P_5$ are to be scheduled in sequence, P_5 necessarily finishing the sequence. Furthermore P_2 has to be scheduled strictly before P_4. P_6 must be completed before or just when P_2 starts, and P_7 is to take place either before or during the execution of P_4. The reason for this is that P_4 is a power-intensive process which causes transients in the power supply for process P_7 just when P_4 is started, and since temperature is critical during this cooling process, this cannot be allowed. Furthermore, P_6 has to start before P_7 starts, and P_6 must not finish before P_6 starts.

In order to optimise the ordering of processes in context with the rest of the production (which is not included here), we would like to allow any ordering consistent with the above constraints. However, the last requirement of P_6 not fininshing before P_6 starts, is not supported by the existing controllers so we would like to verify that this always holds, given all the other constraints, since otherwise, we would need to upgrade the controller.

The example is formalised by letting processes P_i represent interval variables. We obtain the following Allen relations.

$$\begin{array}{llllll} P_1 & (\prec\ \succ) & P_2, & P_1 & (\prec\ \succ) & P_3, \\ P_1 & (\prec\ \succ) & P_4, & P_2 & (\prec\ \succ) & P_3, \\ P_2 & (\prec\ \succ) & P_4, & P_3 & (\prec\ \succ) & P_4, \\ P_2 & (\prec) & P_4, & P_i & (\prec) & P_5 \text{ for all } i \neq 5, \\ P_6 & (\prec\ \mathsf{m}) & P_2, & P_7 & (\prec\ \mathsf{d}) & P_4. \end{array}$$

The condition we would like to verify, is that the constraint

$$P_6(\mathsf{o}\ \mathsf{m})P_7$$

is satisfied in every model. Note that this is equivalent to checking the unsatisfiability of the above constraints together with the constraint

$$P_6(\equiv\ \prec\ \succ\ \mathsf{d}\ \mathsf{d}^{\smile}\ \mathsf{o}^{\smile}\ \mathsf{m}^{\smile}\ \mathsf{s}\ \mathsf{s}^{\smile}\ \mathsf{f}\ \mathsf{f}^{\smile})P_7,$$

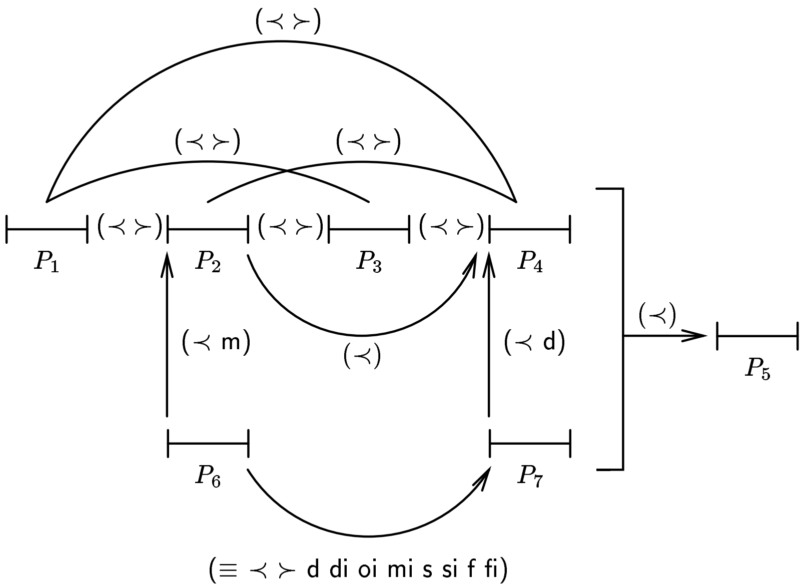

Figure 1: The network describing the constraints on the process.

the complement of the relation (o m), and that all constraints in that network are included in e.g. the tractable algebra $A(m_3, \prec)$ of Proposition 1.1.

We can now show how Algorithm 5.6 performs on this set of constraints (depicted in Figure 1). First, all arcs are already contained in the set

$$A_1(\prec) \cup A_2(m_3, \prec) \cup A_3(m_3, \prec),$$

so nothing needs to be done in the first step. For the second step, the graph G' can be represented by the following set of relations.

$$\begin{array}{llllll} P_2 & (\prec) & P_4, & P_6 & (\prec\ \mathsf{m}) & P_2, \\ P_7 & (\prec\ \mathsf{d}) & P_4, & P_i & (\prec) & P_5 \text{ for all } i \neq 5. \end{array}$$

For the third step, we find all strong components in this graph. It is easy to see that there are no cycles in G', so the set of strong components will be identical to the set of nodes in the graph. Next, we need to check if some relation in the original set of constraints connects two nodes in some component, that is, it is a *loop*, in this case. Obviously, there are no such loops here. Thus, the algorithm accepts, and by correctness, the constraints are satisfiable, and we have no guarantee that our requirement on the process always holds. Consequently, the controller had better be upgraded.

7 The Applicability of Path Consistency

The first attempts at reasoning in Allen's algebra used the *path consistency algorithm* [Montanari, 1974; Mackworth, 1977]; in particular, Allen used such an algorithm [Allen, 1983] as a sound method for checking consistency of interval networks. Also, a recent attempt by Ladkin and Reinefeld [1992] uses the path consistency algorithm as a subroutine in a backtracking search algorithm for reasoning in the full algebra. At each branching step, the algorithm splits disjunctive relations into relations from some algebra for which the path consistency algorithm is complete, thus reducing its branching factor depending on the size of the algebra used. Nebel [1996] proved this method to be correct, and evaluated its efficiency using the ORD-Horn algebra [Nebel and Bürckert, 1995], for which the path consistency algorithm is complete.

In order to make the algebras of this paper useful in such contexts, we prove that the path consistency algorithm is complete for them all. Note, however, that this does not represent an improvement wrt. complexity when using the algebras separately, since path-consistency algorithms typically require at least $O(n^3)$ time [Vilain *et al.*, 1989] where n is the number of interval variables, to be compared to the result of Theorem 5.10, which represents a better running time in the general case.

Definition 7.1 (Path consistent network) Let $V = \{v_1, \ldots, v_m\}$ be interval variables, and $R = \{r_{ij} | 1 \le i,j \le m\}$ Allen relations, where r_{ij} holds between v_i and v_j. Then the network $G = \langle V, R \rangle$ is said to be *path consistent* iff for all v_i, v_j and v_k, it holds that $v_i \circ v_j \supseteq v_k$. $\square$

Definition 7.2 (Path consistency algorithm) Let $V = \{v_1, \ldots, v_m\}$ be interval variables, and $R = \{r_{ij} | 1 \le i,j \le m\}$ Allen relations, where r_{ij} holds between v_i and v_j. The *path consistency algorithm* essentially computes

$$r_{ij} \leftarrow r_{ij} \cap (r_{ik} \circ r_{kj})$$

for all i, j, k until no more changes occur[5]. $\square$

The algorithm can be implemented in a variety of ways, but they are all equivalent to the above definition in terms of input-output behaviour.

[5]The $\leftarrow$ represents *assignment* of r_{ij}.

Corollary 7.3 Let $V = \{v_1, \ldots, v_m\}$ be interval variables, and

$$R = \{r_{ij} | 1 \leq i, j \leq m\}$$

a set of Allen relations, where r_{ij} is the relation between v_i and v_j. Then the result of the path consistency algorithm running on $\langle V, R\rangle$ is path consistent.

Proof: By Definition 7.1 and Definition 7.2. $\Box$

Definition 7.4 (Deciding satisfiability) We say that the path consistency algorithm *decides satisfiability* for a subclass A of Allen's algebra iff for any network $G = \langle V, R\rangle$ of relations from A, if the path consistency algorithm on G is run on G producing the network G', then G is satisfiable iff G' does not contain the relation ( ). $\Box$

Corollary 7.5 The path consistency algorithm decides satisfiability for a subalgebra A of Allen's algebra iff every path consistent network of relations from A not containing ( ) is satisfiable.

Proof: Let $G = \langle V, R\rangle$ be a network of relations from A, and run the path consistency algorithm on G, producing the network G'. Since A is an algebra, G', too, contains only relations from A, by the operation of the algorithm. Now, if G is satisfiable, G' cannot contain ( ), since the algorithm only makes implicit relations explicit (i.e. it is *sound*). If G' does not contain ( ), then G' is satisfiable by the condition, and since G' is more constrained than G, G is also satisfiable. $\Box$

We start by a trivial observation.

Proposition 7.6 The path consistency algorithm decides satisfiability for $A_{\equiv}$.

Proof: If a path consistent network with relations from $A_{\equiv}$ does not contain ( ), then it is satisfiable by setting all intervals equal. The result follows by Corollary 7.5. $\Box$

In order to show the result for the remaining algebras, we first need the following lemmata.

Lemma 7.7 Let $G = \langle V, R\rangle$ be a path consistent network. Then G cannot contain a cycle S with relations only from the set $A = A_2(r, b) \cup A_3(r, b)$, and at least one relation e from $A_2(r, b)$.

Proof: Without loss of generality, suppose that r is one of the four explicitly stated maximal acyclic relations of Proposition 4.8 (i.e. exclude their converses), and assume that such a cycle S exists. Let the interval variables in S be $s_1, s_2, \ldots, s_n$ with relations named $s_i r_{ij} s_j$ for $1 \leq i, j \leq n$. By the construction of A, we have one of the following cases:

1. For any $r \in A$, if IrJ holds in some model M, then $I^- \leq J^-$, and if IeJ holds in some model M, then $I^- < J^-$ holds in M.

2. For any $r \in A$, if IrJ holds in some model M, then $I^+ \leq J^+$, and if IeJ holds in some model M, then $I^+ < J^+$ holds in M.

By symmetry, it is enough to prove the result for the first case.

From path consistency, we get that for any $1 \leq i < k < j \leq n$, $r_{ij} \subseteq r_{ik} \circ r_{kj}$. It is easy to see that by property 1 above, if $r, r' \in A$ and if $Ir \circ r'J$ holds in some model M, then $I^- \leq J^-$ holds in M, and further, by a simple induction and the path consistency property, that for any $1 \leq i < j \leq n$, if $s_i r_{ij} s_j$ holds in some model M, then $s_i^- \leq s_j^-$ holds in M. Suppose, by renumbering the cycle nodes, that $e = r_{n1}$. Thus by property 1 it has to hold that $s_n^- < s_1^-$ in any model of G. But by the above we have that $s_1^- \leq s_n^-$, which is a contradiction. $\Box$

Lemma 7.8 Let $G = \langle V, R \rangle$ be a path consistent network. Then G cannot contain a cycle S with relations only from the set $A = A_3(r, b)$, where two nodes in S are connected by some relation e not containing $(\equiv)$.

Proof: Without loss of generality, suppose that r is one of the four explicitly stated maximal acyclic relations of Proposition 4.8 (i.e. exclude their converses), and assume that such a cycle S exists. Let the interval variables in S be $s_1, s_2, \ldots, s_n$ with relations named $s_i r_{ij} s_j$ for $1 \leq i, j \leq n$. By the construction of A, we have one of the following cases:

1. For any $r \in A$, if IrJ holds in some model M, then $I^- \leq J^-$ and $I^+ \geq J^+$, and if IeJ holds in some model M, then I and J are not identical in M.

2. For any $r \in A$, if IrJ holds in some model M, then $I^+ \leq J^+$ and $I^- \geq J^-$, and if IeJ holds in some model M, then I and J are not identical in M.

By symmetry, it is enough to prove the result for the first case.

From path consistency, we get that for any $1 \leq i < k < j \leq n$, $r_{ij} \subseteq r_{ik} \circ r_{kj}$. It is easy to see that by property 1 above, if $r, r' \in A$ and if

$Ir \circ r'J$ holds in some model M, then $I^- \leq J^-$ and $I^+ \geq J^+$ hold in M, and further, by a simple induction and the path consistency property, that for any $1 \leq i, j \leq n$, if $s_i r_{ij} s_j$ holds in some model M, then $s_i^- \leq s_j^-$ and $s_i^+ \geq s_j^+$ hold in M. But this implies that $r_{ij} \subseteq (\equiv)$ for every relation r_{ij} in the cycle. Let the relation e connect nodes a and b. Now, since e does not contain $\equiv$, $r_{ij} = (\ )$ by path consistency, contradicting the existence of a cycle with the required properties. □

The main result of the section is proved next.

Theorem 7.9 Let $A = A(r, b)$ be one of the algebras from Definition 5.1. Then the path consistency algorithm decides consistency for A.
Proof: Again, we use Corollary 7.5, so suppose that $G = \langle V, R \rangle$ is a path-consistent network of relations from A, not containing $(\ )$, and that G is not satisfiable. Without loss of generality, assume that G only contains relations from $A(b) \cup A_2(r, b) \cup A_3(r, b)$, and consider the graph G', obtained from G by removing arcs which are not labelled by some relation in $A_2(r, b) \cup A_3(r, b)$. By the correctness of Algorithm 5.6, there exists a strong component C of G' and an arc e of G not containing $\equiv$ such that e connects two nodes in C. Suppose that $e \in A_2(r, b)$. Then there is a cycle S of relations in $A_2(r, b) \cup A_3(r, b)$ containing e, which is unsatisfiable. But this is impossible by Lemma 7.7. Thus, suppose that $e \notin A_2(r, b)$ and that C does not contain any relation from $A_2(r, b)$. Then there is a cycle S of relations in $A_3(r, b)$ where two nodes are connected by e. But this is impossible by Lemma 7.8. The result follows by Corollary 7.5. □

8 Discussion

Nebel and Bürckert [1995] argue that the ORD-Horn algebra is an improvement in quantitative terms over previous approaches, since it covers more than 10 percent of the full algebra. Certainly this is a valid argument only because the ORD-Horn algebra *includes* the previous algebras; otherwise we have a counterexample in the $A_\equiv$ algebra, which is much larger than the ORD-Horn algebra, but is clearly of no use. We may mention that the 21 algebras of this paper together cover about 92 percent of $\mathcal{A}$, and that there are only two relations in the ORD-Horn algebra which are not elements of any of the algebras: (m) and $(\mathsf{m}^\smile)$. From a cognitive perspective, the exclusion of these relations is not a serious restriction, as Freksa [1992] notes, since they are not likely to occur in any context reasoning about, for instance, perception of the physical world.

It is also argued by Nebel and Bürckert [1995] that a useful algebra should contain all the basic relations, since otherwise, complete knowledge cannot be specified. However, since the unique maximality of the ORD-Horn class shows that there exists no tractable subalgebra which contains both all the basic relations and the relations expressing *sequentiality* (notably the $(\prec\ \succ)$ relation), this argument fails. Furthermore, four of our algebras can indeed express this sequentiality requirement, which underlies many systems (see e.g. [Sandewall, 1994]).

Given four such algebras, a justified question is whether these are just four out of, for example, four hundred such tractable classes, that is, what makes these algebras relevant? Fortunately, recent research [Drakengren and Jonsson, 1997b] has shown that the two maximal tractable extensions of these four algebras found by Drakengren and Jonsson [1997a] are the *only* maximal tractable algebras capable of expressing sequentiality. Similarly, in the point-interval algebra, we have only five maximal tractable subclasses [Jonsson *et al.*, 1996] indicating that maximal tractable algebras are typically scarce.

Also, as a long-term goal, it would be useful to classify *all* maximal tractable subalgebras of the full algebra, since then an application using networks of interval relations could search for the best algebra to use, or otherwise report that no such algebra exists. Since there are 2^{8192} subsets of the full algebra, the task is clearly nontrivial, even using computer-supported proof methods. Our work with Bäckström on the point-interval algebra [Jonsson *et al.*, 1996] and our recent partial classification of the Allen algebra [Drakengren and Jonsson, 1997b] show that a complete classification might not be out of reach.

Recently, a proof method for the Allen algebra has been presented by Ligozat [1996], which was applied to find a new tractability proof for satisfiability in the ORD-Horn algebra. Even though the method seems to be quite general, it is not applicable to the algebras of this paper, since the *convexity* properties which are crucial are not satisfied, notably by relations like $(\prec\ \succ)$.

Finally, it is interesting to note that although we have an algorithm for satisfiability of all the algebras, we cannot automatically conclude that the problem of *entailment*, which is that of deciding what relation is induced between two intervals, is solved: in the case when a tractable algebra contains all of the basic relations, this problem reduces to satisfiability, as shown by Golumbic and Shamir [1993], but this is not at all obvious when that restriction is not satisfied.

9 Conclusions

We have identified 21 new large tractable fragments of Allen's interval algebra, of which nine have been proved maximal tractable. Further, we have presented a linear time algorithm for deciding satisfiability of these. In addition, all the algebras are considerably larger (in quantity) than the ORD-Horn subalgebra, but thus cannot contain all the basic relations. Also, four of the algebras can express the relations $(\equiv)$, $(\prec)$, $(\prec\ \equiv)$, $(\succ)$, $(\succ\ \equiv)$ and $(\prec\ \succ)$ (in addition to the "nonrelation"), which is necessary and sufficient for expressing the notion of sequentiality. Finally, we showed that the path consistency algorithm decides satisfiability for all of the algebras.

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Appendix

1 An Explicit Listing of the Algebras

Proposition 1.1 First abbreviate the maximal acyclic relations by

$$\begin{array}{rcl} m_1 & = & (\prec\ \mathsf{d}^\smile\ \mathsf{o}\ \mathsf{m}\ \mathsf{s}\ \mathsf{f}^\smile) \\ m_2 & = & (\prec\ \mathsf{d}^\smile\ \mathsf{o}\ \mathsf{m}\ \mathsf{s}^\smile\ \mathsf{f}^\smile) \\ m_3 & = & (\prec\ \mathsf{d}\ \mathsf{o}\ \mathsf{m}\ \mathsf{s}\ \mathsf{f}) \\ m_4 & = & (\prec\ \mathsf{d}\ \mathsf{o}\ \mathsf{m}\ \mathsf{s}\ \mathsf{f}^\smile), \end{array}$$

obtaining the remaining ones by taking the converses of these.

Now $A(m_1, \prec)$ is defined by

$$\begin{array}{lclcl} (\prec\ \succ) & \subseteq & r' & & \\ (\prec) & \subseteq & r' & \subseteq & (\equiv\ \prec\ \mathsf{d}^\smile\ \mathsf{o}\ \mathsf{m}\ \mathsf{s}\ \mathsf{f}^\smile) \\ (\succ) & \subseteq & r' & \subseteq & (\equiv\ \succ\ \mathsf{d}\ \mathsf{o}^\smile\ \mathsf{m}^\smile\ \mathsf{s}^\smile\ \mathsf{f}) \\ & & r' & \subseteq & (\equiv), \end{array}$$

$A(m_2, \prec)$ by

$$\begin{array}{lclcl} (\prec\ \succ) & \subseteq & r' & & \\ (\prec) & \subseteq & r' & \subseteq & (\equiv\ \prec\ \mathsf{d}^\smile\ \mathsf{o}\ \mathsf{m}\ \mathsf{s}^\smile\ \mathsf{f}^\smile) \\ (\succ) & \subseteq & r' & \subseteq & (\equiv\ \succ\ \mathsf{d}\ \mathsf{o}^\smile\ \mathsf{m}^\smile\ \mathsf{s}\ \mathsf{f}) \\ & & r' & \subseteq & (\equiv), \end{array}$$

$A(m_3, \prec)$ by

$$\begin{array}{lclcl} (\prec\ \succ) & \subseteq & r' & & \\ (\prec) & \subseteq & r' & \subseteq & (\equiv\ \prec\ \mathsf{d}\ \mathsf{o}\ \mathsf{m}\ \mathsf{s}\ \mathsf{f}) \\ (\succ) & \subseteq & r' & \subseteq & (\equiv\ \succ\ \mathsf{d}^\smile\ \mathsf{o}^\smile\ \mathsf{m}^\smile\ \mathsf{s}^\smile\ \mathsf{f}^\smile) \\ & & r' & \subseteq & (\equiv), \end{array}$$

$A(m_4, \prec)$ by

$$\begin{array}{lclcl} (\prec\ \succ) & \subseteq & r' & & \\ (\prec) & \subseteq & r' & \subseteq & (\equiv\ \prec\ \mathsf{d}\ \mathsf{o}\ \mathsf{m}\ \mathsf{s}\ \mathsf{f}^\smile) \\ (\succ) & \subseteq & r' & \subseteq & (\equiv\ \succ\ \mathsf{d}^\smile\ \mathsf{o}^\smile\ \mathsf{m}^\smile\ \mathsf{s}^\smile\ \mathsf{f}) \\ & & r' & \subseteq & (\equiv), \end{array}$$

$A({m_1}^\smile, \mathsf{d})$ by

$$\begin{array}{lllll}
(\mathsf{d}\ \mathsf{d}^{\smile}) & \subseteq & r' & & \\
(\mathsf{d}) & \subseteq & r' & \subseteq & (\equiv\ \succ\ \mathsf{d}\ \mathsf{o}^{\smile}\ \mathsf{m}^{\smile}\ \mathsf{s}^{\smile}\ \mathsf{f}) \\
(\mathsf{d}^{\smile}) & \subseteq & r' & \subseteq & (\equiv\ \prec\ \mathsf{d}^{\smile}\ \mathsf{o}\ \mathsf{m}\ \mathsf{s}\ \mathsf{f}^{\smile}) \\
 & & r' & \subseteq & (\equiv),
\end{array}$$

$A(m_2{}^{\smile}, \mathsf{d})$ by

$$\begin{array}{lllll}
(\mathsf{d}\ \mathsf{d}^{\smile}) & \subseteq & r' & & \\
(\mathsf{d}) & \subseteq & r' & \subseteq & (\equiv\ \succ\ \mathsf{d}\ \mathsf{o}^{\smile}\ \mathsf{m}^{\smile}\ \mathsf{s}\ \mathsf{f}) \\
(\mathsf{d}^{\smile}) & \subseteq & r' & \subseteq & (\equiv\ \prec\ \mathsf{d}^{\smile}\ \mathsf{o}\ \mathsf{m}\ \mathsf{s}^{\smile}\ \mathsf{f}^{\smile}) \\
 & & r' & \subseteq & (\equiv),
\end{array}$$

$A(m_3, \mathsf{d})$ by

$$\begin{array}{lllll}
(\mathsf{d}\ \mathsf{d}^{\smile}) & \subseteq & r' & & \\
(\mathsf{d}) & \subseteq & r' & \subseteq & (\equiv\ \prec\ \mathsf{d}\ \mathsf{o}\ \mathsf{m}\ \mathsf{s}\ \mathsf{f}) \\
(\mathsf{d}^{\smile}) & \subseteq & r' & \subseteq & (\equiv\ \succ\ \mathsf{d}^{\smile}\ \mathsf{o}^{\smile}\ \mathsf{m}^{\smile}\ \mathsf{s}^{\smile}\ \mathsf{f}^{\smile}) \\
 & & r' & \subseteq & (\equiv),
\end{array}$$

$A(m_4, \mathsf{d})$ by

$$\begin{array}{lllll}
(\mathsf{d}\ \mathsf{d}^{\smile}) & \subseteq & r' & & \\
(\mathsf{d}) & \subseteq & r' & \subseteq & (\equiv\ \prec\ \mathsf{d}\ \mathsf{o}\ \mathsf{m}\ \mathsf{s}\ \mathsf{f}^{\smile}) \\
(\mathsf{d}^{\smile}) & \subseteq & r' & \subseteq & (\equiv\ \succ\ \mathsf{d}^{\smile}\ \mathsf{o}^{\smile}\ \mathsf{m}^{\smile}\ \mathsf{s}^{\smile}\ \mathsf{f}) \\
 & & r' & \subseteq & (\equiv),
\end{array}$$

$A(m_1, \mathsf{o})$ by

$$\begin{array}{lllll}
(\mathsf{o}\ \mathsf{o}^{\smile}) & \subseteq & r' & & \\
(\mathsf{o}) & \subseteq & r' & \subseteq & (\equiv\ \prec\ \mathsf{d}^{\smile}\ \mathsf{o}\ \mathsf{m}\ \mathsf{s}\ \mathsf{f}^{\smile}) \\
(\mathsf{o}^{\smile}) & \subseteq & r' & \subseteq & (\equiv\ \succ\ \mathsf{d}\ \mathsf{o}^{\smile}\ \mathsf{m}^{\smile}\ \mathsf{s}^{\smile}\ \mathsf{f}) \\
 & & r' & \subseteq & (\equiv),
\end{array}$$

$A(m_2, \mathsf{o})$ by

$$\begin{array}{lllll}
(\mathsf{o}\ \mathsf{o}^{\smile}) & \subseteq & r' & & \\
(\mathsf{o}) & \subseteq & r' & \subseteq & (\equiv\ \prec\ \mathsf{d}^{\smile}\ \mathsf{o}\ \mathsf{m}\ \mathsf{s}^{\smile}\ \mathsf{f}^{\smile}) \\
(\mathsf{o}^{\smile}) & \subseteq & r' & \subseteq & (\equiv\ \succ\ \mathsf{d}\ \mathsf{o}^{\smile}\ \mathsf{m}^{\smile}\ \mathsf{s}\ \mathsf{f}) \\
 & & r' & \subseteq & (\equiv),
\end{array}$$

$A(m_3, \mathsf{o})$ by

$$\begin{array}{llllll}
(\mathsf{o}\ \mathsf{o}^{\smile}) & \subseteq & r' & & \\
(\mathsf{o}) & \subseteq & r' & \subseteq & (\equiv\ \prec\ \ \mathsf{d}\ \mathsf{o}\ \mathsf{m}\ \mathsf{s}\ \mathsf{f}) \\
(\mathsf{o}^{\smile}) & \subseteq & r' & \subseteq & (\equiv\ \succ\ \ \mathsf{d}^{\smile}\ \mathsf{o}^{\smile}\ \mathsf{m}^{\smile}\ \mathsf{s}^{\smile}\ \mathsf{f}^{\smile}) \\
 & & r' & \subseteq & (\equiv),
\end{array}$$

$A(m_4, \mathsf{o})$ by

$$\begin{array}{llllll}
(\mathsf{o}\ \mathsf{o}^{\smile}) & \subseteq & r' & & \\
(\mathsf{o}) & \subseteq & r' & \subseteq & (\equiv\ \prec\ \ \mathsf{d}\ \mathsf{o}\ \mathsf{m}\ \mathsf{s}\ \mathsf{f}^{\smile}) \\
(\mathsf{o}^{\smile}) & \subseteq & r' & \subseteq & (\equiv\ \succ\ \ \mathsf{d}^{\smile}\ \mathsf{o}^{\smile}\ \mathsf{m}^{\smile}\ \mathsf{s}^{\smile}\ \mathsf{f}) \\
 & & r' & \subseteq & (\equiv),
\end{array}$$

$A(m_1, \mathsf{s})$ by

$$\begin{array}{llllll}
(\mathsf{s}\ \mathsf{s}^{\smile}) & \subseteq & r' & & \\
(\mathsf{s}) & \subseteq & r' & \subseteq & (\equiv\ \prec\ \ \mathsf{d}^{\smile}\ \mathsf{o}\ \mathsf{m}\ \mathsf{s}\ \mathsf{f}^{\smile}) \\
(\mathsf{s}^{\smile}) & \subseteq & r' & \subseteq & (\equiv\ \succ\ \ \mathsf{d}\ \mathsf{o}^{\smile}\ \mathsf{m}^{\smile}\ \mathsf{s}^{\smile}\ \mathsf{f}) \\
 & & r' & \subseteq & (\equiv),
\end{array}$$

$A(m_2{}^{\smile}, \mathsf{s})$ by

$$\begin{array}{llllll}
(\mathsf{s}\ \mathsf{s}^{\smile}) & \subseteq & r' & & \\
(\mathsf{s}) & \subseteq & r' & \subseteq & (\equiv\ \succ\ \ \mathsf{d}\ \mathsf{o}^{\smile}\ \mathsf{m}^{\smile}\ \mathsf{s}\ \mathsf{f}) \\
(\mathsf{s}^{\smile}) & \subseteq & r' & \subseteq & (\equiv\ \prec\ \ \mathsf{d}^{\smile}\ \mathsf{o}\ \mathsf{m}\ \mathsf{s}^{\smile}\ \mathsf{f}^{\smile}) \\
 & & r' & \subseteq & (\equiv),
\end{array}$$

$A(m_3, \mathsf{s})$ by

$$\begin{array}{llllll}
(\mathsf{s}\ \mathsf{s}^{\smile}) & \subseteq & r' & & \\
(\mathsf{s}) & \subseteq & r' & \subseteq & (\equiv\ \prec\ \ \mathsf{d}\ \mathsf{o}\ \mathsf{m}\ \mathsf{s}\ \mathsf{f}) \\
(\mathsf{s}^{\smile}) & \subseteq & r' & \subseteq & (\equiv\ \succ\ \ \mathsf{d}^{\smile}\ \mathsf{o}^{\smile}\ \mathsf{m}^{\smile}\ \mathsf{s}^{\smile}\ \mathsf{f}^{\smile}) \\
 & & r' & \subseteq & (\equiv),
\end{array}$$

$A(m_4, \mathsf{s})$ by

$$\begin{array}{llllll}
(\mathsf{s}\ \mathsf{s}^{\smile}) & \subseteq & r' & & \\
(\mathsf{s}) & \subseteq & r' & \subseteq & (\equiv\ \prec\ \ \mathsf{d}\ \mathsf{o}\ \mathsf{m}\ \mathsf{s}\ \mathsf{f}^{\smile}) \\
(\mathsf{s}^{\smile}) & \subseteq & r' & \subseteq & (\equiv\ \succ\ \ \mathsf{d}^{\smile}\ \mathsf{o}^{\smile}\ \mathsf{m}^{\smile}\ \mathsf{s}^{\smile}\ \mathsf{f}) \\
 & & r' & \subseteq & (\equiv),
\end{array}$$

$A(m_1{}^{\smile}, \mathsf{f})$ by

$$\begin{array}{llllll}
(\mathsf{f}\ \mathsf{f}^\smile) & \subseteq & r' & & \\
(\mathsf{f}) & \subseteq & r' & \subseteq & (\equiv\ \succ\ \ \mathsf{d}\ \mathsf{o}^\smile\ \mathsf{m}^\smile\ \mathsf{s}^\smile\ \mathsf{f}) \\
(\mathsf{f}^\smile) & \subseteq & r' & \subseteq & (\equiv\ \prec\ \ \mathsf{d}^\smile\ \mathsf{o}\ \mathsf{m}\ \mathsf{s}\ \mathsf{f}^\smile) \\
 & & r' & \subseteq & (\equiv),
\end{array}$$

$A(m_2{}^\smile, \mathsf{f})$ by

$$\begin{array}{llllll}
(\mathsf{f}\ \mathsf{f}^\smile) & \subseteq & r' & & \\
(\mathsf{f}) & \subseteq & r' & \subseteq & (\equiv\ \succ\ \ \mathsf{d}\ \mathsf{o}^\smile\ \mathsf{m}^\smile\ \mathsf{s}\ \mathsf{f}) \\
(\mathsf{f}^\smile) & \subseteq & r' & \subseteq & (\equiv\ \prec\ \ \mathsf{d}^\smile\ \mathsf{o}\ \mathsf{m}\ \mathsf{s}^\smile\ \mathsf{f}^\smile) \\
 & & r' & \subseteq & (\equiv),
\end{array}$$

$A(m_3, \mathsf{f})$ by

$$\begin{array}{llllll}
(\mathsf{f}\ \mathsf{f}^\smile) & \subseteq & r' & & \\
(\mathsf{f}) & \subseteq & r' & \subseteq & (\equiv\ \prec\ \ \mathsf{d}\ \mathsf{o}\ \mathsf{m}\ \mathsf{s}\ \mathsf{f}) \\
(\mathsf{f}^\smile) & \subseteq & r' & \subseteq & (\equiv\ \succ\ \ \mathsf{d}^\smile\ \mathsf{o}^\smile\ \mathsf{m}^\smile\ \mathsf{s}^\smile\ \mathsf{f}^\smile) \\
 & & r' & \subseteq & (\equiv),
\end{array}$$

and finally, $A(m_4{}^\smile, \mathsf{f})$ by

$$\begin{array}{llllll}
(\mathsf{f}\ \mathsf{f}^\smile) & \subseteq & r' & & \\
(\mathsf{f}) & \subseteq & r' & \subseteq & (\equiv\ \succ\ \ \mathsf{d}^\smile\ \mathsf{o}^\smile\ \mathsf{m}^\smile\ \mathsf{s}^\smile\ \mathsf{f}) \\
(\mathsf{f}^\smile) & \subseteq & r' & \subseteq & (\equiv\ \prec\ \ \mathsf{d}\ \mathsf{o}\ \mathsf{m}\ \mathsf{s}\ \mathsf{f}^\smile) \\
 & & r' & \subseteq & (\equiv),
\end{array}$$

The remaining combinations, which are redundant, are obtained by the equation

$$A(r, b) = A(r^\smile, b^\smile).$$

□

Paper III

Eight Maximal Tractable Subclasses of Allen's Algebra with Metric Time

Thomas Drakengren and Peter Jonsson

ABSTRACT

This paper combines two important directions of research in temporal resoning: that of finding maximal tractable subclasses of Allen's interval algebra, and that of reasoning with metric temporal information. Eight new maximal tractable subclasses of Allen's interval algebra are presented, some of them subsuming previously reported tractable algebras. The algebras allow for metric temporal constraints on interval starting or ending points, using the recent framework of *Horn DLRs*. Two of the algebras can express the notion of *sequentiality* between intervals, being the first such algebras admitting both qualitative and metric time.

1 Introduction

Reasoning about temporal knowledge abounds in artificial intelligence applications and other areas, such as planning [Allen, 1991], natural language understanding [Song and Cohen, 1988] and molecular biology [Benzer, 1959; Golumbic and Shamir, 1993]. However, since even the restricted problem of reasoning with pure qualitative time in Allen's interval algebra [Allen, 1983] is NP-complete [Vilain *et al.*, 1989], research has focused on identifying classes of problems where reasoning is tractable [Drakengren and Jonsson, 1996; Golumbic and Shamir, 1993; Kautz and Ladkin, 1991; Nebel and Bürckert, 1995; van Beek and Cohen, 1990; van Beek, 1989; van Beek, 1992].

Until recently, approaches have mostly been either metric or qualitative, with a few exceptions [Kautz and Ladkin, 1991; Meiri, 1991; Gerevini *et al.*, 1993]. However, the approach of Jonsson and Bäckström [1996] (also developed independently by Koubarakis, 1996) manages to unify almost every approach to tractable reasoning about metric time, qualitative time and the integrated approaches in one framework, that of *Horn disjunctive linear relations* (Horn DLRs), in which the reasoning problem can be solved in polynomial time. Since its expressiveness in terms of qualitative information subsumes that of the maximal tractable[1] ORD-Horn algebra [Nebel and Bürckert, 1995], it can be viewed as a maximal tractable subalgebra of Allen's interval algebra, provided with metric temporal information.

First, this paper continues the work on finding maximal tractable subalgebras of Allen's algebra started by Nebel and Bürckert [1995] and continued by Drakengren and Jonsson [1996], by identifying eight more maximal tractable subclasses of Allen's algebra. Second, we combine this with the work of Jonsson and Bäckström [1996], by providing the maximal algebras with metric temporal information in the form of Horn DLRs, whose expressiveness subsumes that of the ORD-Horn algebra. Further, the maximality result of these algebras settles the question of maximality of some algebras in Drakengren and Jonsson [1996], since these are included in the new algebras presented here. Two of the maximal algebras can express the notion of *sequentiality* between intervals[2], important for example in reasoning about action [Sandewall, 1994], where actions are typically assumed to occur in sequence. To our knowledge, these are the first such algebras in the literature

[1] That is, it is tractable, and no other tractable algebra strictly contains it.

[2] That is, they contain the relations $(\prec)$, $(\succ)$, $(\prec\ \succ)$, $(\equiv\ \prec)$, $(\equiv\ \succ)$, $(\equiv\ \prec\ \succ)$, and the "unrelated" relation, and can thus express that "two intervals occur in sequence".

Basic relation		Example	Endpoints
x before y	$\prec$	xxx	$x^+ < y^-$
y after x	$\succ$	yyy	
x meets y	m	xxxx	$x^+ = y^-$
y met-by x	$\mathsf{m}^\smile$	yyyy	
x overlaps y	o	xxxx	$x^- < y^- < x^+$,
y overl.-by x	$\mathsf{o}^\smile$	yyyy	$x^+ < y^+$
x during y	d	xxx	$x^- > y^-$,
y includes x	$\mathsf{d}^\smile$	yyyyyyy	$x^+ < y^+$
x starts y	s	xxx	$x^- = y^-$,
y started by x	$\mathsf{s}^\smile$	yyyyyyy	$x^+ < y^+$
x finishes y	f	xxx	$x^+ = y^+$,
y finished by x	$\mathsf{f}^\smile$	yyyyyyy	$x^- > y^-$
x equals y	$\equiv$	xxxx	$x^- = y^-$,
		yyyy	$x^+ = y^+$

Table 1: The thirteen basic relations. The endpoint relations $x^- < x^+$ and $y^- < y^+$ that are valid for all relations have been omitted.

that are also provided with metric temporal information. By the fact that we can also relate starting or ending points by Horn DLRs, we have a combination (with restrictions, of course) of the expressiveness of the ORD-Horn algebra, and that of sequentiality.

The structure of the paper is as follows. Section 2 introduces Allen's interval algebra, Section 3 defines the concepts needed to integrate qualitative and metric reasoning, together with the satisfiability algorithm, and Section 4 presents the new maximal tractable algebras, and how to provide these with metric temporal information. In Section 5 and Section 6, we prove that the algorithm is correct and that the algebras are maximal. Finally, Section 7 and Section 8 discuss and conclude the paper.

2 Allen's Interval Algebra

Allen's interval algebra [Allen, 1983] is based on the notion of *relations between pairs of intervals*. An interval x is represented as a tuple $\langle x^-, x^+\rangle$ of real numbers with $x^- < x^+$, denoting the left and right endpoints of the interval, respectively, and relations between intervals are composed as disjunctions of *basic interval relations*, which are those in Table 1. Such disjunctions are represented as *sets* of basic relations, but using a notation such that, for example, the disjunction of the basic intervals $\prec$, m and $\mathsf{f}^{\smile}$ is written $(\prec\ \mathsf{m}\ \mathsf{f}^{\smile})$. Thus, we have that $(\prec\ \mathsf{f}^{\smile}) \subseteq (\prec\ \mathsf{m}\ \mathsf{f}^{\smile})$. Sometimes the disjunction of *all* basic relations is written $\top$ and the empty relation is written $\perp$ (this is also used for relations between interval endpoints, denoting "always satisfiable" and "unsatisfiable", respectively). The algebra is provided with the operations of *converse, intersection and composition* on intervals, but we shall need only the converse operation explicitly. The converse operation takes an interval relation i to its converse $i^{\smile}$, obtained by inverting each basic relation in i, i.e., exchanging x and y in the endpoint relations shown in Table 1.

By the fact that there are thirteen basic relations, we get $2^{13} = 8192$ possible relations between intervals in the full algebra. We denote the set of all interval relations by $\mathcal{A}$. Subclasses of the full algebra are obtained by considering subsets of $\mathcal{A}$. There are $2^{8192} \approx 10^{2466}$ such subclasses. Classes that are closed under the operations of intersection, converse and composition are said to be *algebras*.

Although there are several problems of computation associated with Allen's interval algebra, this paper focuses on the problem of *satisfiability* (*ISAT*) of a set of interval variables with relations between them, i.e., deciding whether there exists an assignment of intervals on the real line for the interval variables, such that all of the relations between the intervals are satisfied. We define this as follows.

Definition 2.1 **(*ISAT*$(\mathcal{I})$)** Let $\mathcal{I}$ be a set of interval relations. An instance of *ISAT*$(\mathcal{I})$ is a labelled directed graph $S = \langle V, E\rangle$, where the nodes in V are interval variables and E is a subset of $V \times \mathcal{I} \times V$. A labelled edge $\langle u, r, v\rangle \in E$ means that u and v are related by r.

A function M taking an interval variable v to its interval representation $M(v) = \langle x^-, x^+\rangle$ with $x^-, x^+ \in \mathbb{R}$ and $x^- < x^+$, is said to be an *interpretation* of S.

An instance $S = \langle V, E\rangle$ is said to be *satisfiable* iff there exists an interpretation M such that for each $\langle u, r, v\rangle \in E$, $M(u)rM(v)$ holds, i.e., the endpoint relations required by r (see Table 1) are satisfied by the assignments of u and v. Then M is said to be a *model* of S.

We refer to the *size* of an instance S as $|V| + |E|$. $\square$

For $\mathcal{A}$, we have the following result.

Proposition 2.2 *ISAT*$(\mathcal{A})$ is NP-complete.
Proof: See Vilain et al. [1989]. □

3 Qualitative and Metric Time

We first briefly recapitulate the Horn DLR formalism of Jonsson and Bäckström [1996].

Definition 3.1 (Linear relation, Disjunctive linear relation) Let $X = \{x_1, \ldots, x_n\}$ be a set of real-valued variables, and α, β linear polynomials (polynomials of degree one) over X with rational coefficients. A *linear relation* over X is a mathematical expression of the form $\alpha r \beta$, where $r \in \{<, \leq, =, \neq, \geq, >\}$. A *disjunctive linear relation* (DLR) over X is a disjunction of one or more linear relations. A DLR is said to be *Horn* iff at most one of its disjuncts is not of the form $\alpha \neq \beta$.

The satisfiability problem for finite sets H of Horn DLRs is denoted HORNDLRSAT(H), checking whether there exists an assignment M of variables in X to real numbers such that all DLRs in H are satisfied in M. Such an M is said to be a *model* of H. □

Example 3.2 $x + 2y \leq 3z + 42.3$
is a linear relation,

$$(x + 2y \leq 3z + 42.3) \vee (x > \frac{3}{12})$$

is a disjunctive linear relation, and

$$(x + 2y \leq 3z + 42.3) \vee (x \neq \frac{3}{12})$$

is a Horn disjunctive linear relation. □

Proposition 3.3 There is a polynomial-time algorithm for HORNDLRSAT(H).

Proof: See Jonsson and Bäckström [1996] or Koubarakis [1996]. □

In principle, the framework of DLRs makes it unnecessary to distinguish between qualitative and metric information. Nevertheless, when it comes to identifying tractable subclasses, the distinction is still convenient.

The Horn DLR approach subsumes almost all previously known approaches to tractable metric and qualitative temporal reasoning, e.g. [Nebel and Bürckert, 1995; Koubarakis, 1992; Dechter *et al.*, 1991; Meiri, 1991; Gerevini *et al.*, 1993]. It is worth mentioning that the maximal tractable algebras found by Drakengren and Jonsson [1996] cannot be expressed as Horn DLRs.

Although polynomial, the algorithm presented in Jonsson and Bäckström [1996] is quite expensive (it relies on a linear-programming algorithm) so when we have no need of specifying metric information, the following well-known subclass of the set of Horn DLRs will result in a lower-complexity algorithm.

Definition 3.4 (The point algebra) The *point algebra* [Vilain, 1982] is the subclass of Horn DLRs consisting of the set of expressions xRy, where x and y are variables, and R is one of the relations $<$, $\leq$, $=$, $\neq$, $\geq$ and $>$. □

The satisfiability problem for this subclass is denoted $\mathrm{PA}\textsc{sat}(H)$, for a set H of point algebra formulae.

Proposition 3.5 $\mathrm{PA}\textsc{sat}(H)$ is solvable in linear time in the size of H.
Proof: See Gerevini et al. [1993] (for practical purposes the algorithm of Delgrande and Gupta, 1996, could be preferred). □

Next, we define the problem of interest in this paper — the interval satisfiability problem with metric temporal information.

Definition 3.6 (M-$ISAT(\mathcal{I})$) Let $\langle V, E\rangle$ be an instance of $ISAT(\mathcal{I})$ and H a finite set of DLRs over the set $\{v^+, v^- \mid v \in V\}$ of variables, v^- representing starting points and v^+ ending points of intervals v.

An instance of the problem of *interval satisfiability with metric information* for a set $\mathcal{I}$ of interval relations, denoted M-$ISAT(\mathcal{I})$, is a tuple $Q = \langle V, E, H\rangle$.

An *interpretation* M for Q is an interpretation for $\langle V, E\rangle$. Since we now need to refer to starting and ending points of intervals, we extend the notation such that $M(v^-)$ obtains the starting point of the interval $M(v)$, and similarly for $M(v^+)$.

An instance Q is said to be *satisfiable* iff there exists a model M of $\langle V, E\rangle$ such that the DLRs in H are satisfied, with values for all v^- and v^+ by $M(v^-)$ and $M(v^+)$, respectively. □

Since every Allen interval relation can be expressed as a DLR (but not necessarily as a Horn DLR), we could instead have formulated the problem as a pure satisfiability problem of a set of DLRs, but since we are interested in the particular structure imposed on the problem by interval relations specifically we prefer this formulation.

Several concepts are needed in order to present the *starting and ending point algebras*, for which we shall provide polynomial-time algorithms. The curious reader might temporarily jump to Section 4 for the explicit presentation of the algebras which will be proved to be starting or ending points algebras.

The following definitions are needed to transfer information from interval relations to point relations.

Definition 3.7 **(**$\mathit{sprel}(r)$**,** $\mathit{eprel}(r)$**,** $\mathit{sprel}^+(r)$**,** $\mathit{eprel}^-(r)$**)** Take the relation $r \in \mathcal{A}$, let u and v be interval variables, and consider the instance S of $\mathit{ISAT}(\{r\})$ which relates u and v with the relation r only. Define the relation $\mathit{sprel}(r)$ on real numbers to be the symbol for the implied relation between the starting points of u and v. That is, for basic relations define

$$\begin{aligned}
\mathit{sprel}(\equiv) &= \text{“}{=}\text{”} \\
\mathit{sprel}(\prec) &= \text{“}{<}\text{”} \\
\mathit{sprel}(\mathsf{d}) &= \text{“}{>}\text{”} \\
\mathit{sprel}(\mathsf{o}) &= \text{“}{<}\text{”} \\
\mathit{sprel}(\mathsf{m}) &= \text{“}{<}\text{”} \\
\mathit{sprel}(\mathsf{s}) &= \text{“}{=}\text{”} \\
\mathit{sprel}(\mathsf{f}) &= \text{“}{>}\text{”} \\
\mathit{sprel}(r^{\smile}) &= (\mathit{sprel}(r))^{-1},
\end{aligned}$$

and for disjunctions of basic relations $\mathit{sprel}(r)$ is the relation symbol corresponding to $\bigvee_{b \in r} \mathit{sprel}(b)$. For example, $\mathit{sprel}((\prec\ \succ)) = \text{“}{\neq}\text{”}$. Symmetrically, we define $\mathit{eprel}(r)$ to be the implied relation between ending points given r. Note that $\mathit{sprel}(r)$ and $\mathit{eprel}(r)$ have to be either of $<$, $\leq$, $=$, $\geq$, $>$, $\neq$, $\top$ or $\bot$.

Further, we define specialisations of these, by setting

$$\mathit{sprel}^+(r) = \mathit{sprel}(r \cap (\equiv\ \ \mathsf{f}\ \mathsf{f}^{\smile}))$$

and

$$\mathit{eprel}^-(r) = \mathit{eprel}(r \cap (\equiv\ \ \mathsf{s}\ \mathsf{s}^{\smile})),$$

i.e., the implied relations on starting (ending) points by r, given that the ending (starting) points are known to be equal. $\square$

Definition 3.8 (Explicit starting (ending) point relations) Let $\mathcal{I} \subseteq \mathcal{A}$, $Q = \langle V, E, H \rangle$ an instance of $M\text{-}ISAT(\mathcal{I})$, and construct the instance $Q' = \langle V, E, H' \rangle$ of $M\text{-}ISAT(\mathcal{I})$ by setting

$$H' = H \cup \{u^- \mathit{sprel}(r) v^- \mid \langle u, r, v \rangle \in E\}.$$

Then Q' is said to be obtained from Q by *making starting point relations explicit*. We denote this Q' by $\mathit{expl}^-(Q)$.

Symmetrically, using *eprel* and ending points instead of *sprel* and starting points, Q' is said to be obtained from Q by *making ending points explicit*, denoted $\mathit{expl}^+(Q)$. $\square$

It is easy to see that only point algebra formulae are added to H.

Transferring information from interval relations to point relations does not change satisfiability, as expected:

Proposition 3.9 Let $\mathcal{I} \subseteq \mathcal{A}$ and Q an instance of $M\text{-}ISAT(\mathcal{I})$. Then Q is satisfiable iff $\mathit{expl}^-(Q)$ is satisfiable iff $\mathit{expl}^+(Q)$ is satisfiable.
Proof: By the fact that the added starting and ending point relations are already guaranteed to hold in any model of Q. $\square$

We jump ahead by presenting a satisfiability algorithm (Algorithm 3.10) and briefly discuss the intuition behind it in order to indicate what kind of algebras it works for. This will hopefully make it easier to appreciate Definition 3.13.

First assume that H only contains Horn DLRs, which only relate starting points of intervals. Line 1 makes the interval relations explicit as starting point relations and line 2 checks satisfiability of the resulting set of starting point relations. Lines 4 to 11 collect in K the relations $u^- = v^-$, such that in any model these starting points have to be equal. In addition, K forces all starting points to be distinct, that are not forced to be equal. It is clear that the equality formulae in K do not affect satisfiability. However, it is less clear that the disequality formulae in K cannot make the instance $Q'' = \langle V, E, H' \cup K \rangle$ unsatisfiable. This fact indeed follows from a property of Horn DLRs, which is proved in Theorem 5.9. At line 12, we know that there are no two models for Q'', where for some $u, v \in V$, $u^- = v^-$ in one model, and $u^- \neq v^-$ in the other model. This is the intuition behind Definition 3.11. Now, line 13 checks for satisfiability of the ending points of those intervals whose starting points have to be equal in any model. If the algorithm rejects at line 14, then the instance is obviously not satisfiable. Otherwise we need

Algorithm 3.10 (M_s-*ISAT*$(\mathcal{I})$)

input Instance $Q = \langle V, E, H \rangle$

1 $Q' = \langle V, E, H' \rangle \leftarrow expl^{-}(Q)$
2 **if** not HORNDLRSAT(H') **then**
3   **reject**
4 $K \leftarrow \emptyset$
5 **for** each $\langle u, r, v \rangle \in E$
6   **if** not HORNDLRSAT$(H' \cup \{u^- \neq v^-\})$ **then**
7       $K \leftarrow K \cup \{u^- = v^-\}$
8   **else**
9       $K \leftarrow K \cup \{u^- \neq v^-\}$
10   **endif**
11 **endfor**
12 $P \leftarrow \{u^+ eprel^-(r) v^+ \mid \langle u, r, v \rangle \in E \wedge u^- = v^- \in H' \cup K\}$
13 **if** not PASAT(P) **then**
14   **reject**
15 **accept**

□

a condition on the algebra $\mathcal{I}$, corresponding to Definition 3.13, in order to guarantee satisfiability.

The formal machinery follows.

Definition 3.11 (Starting (ending) point definite) Let $\mathcal{I} \subseteq \mathcal{A}$, and $Q = \langle V, E, H \rangle$ an instance of M-*ISAT*$(\mathcal{I})$. The instance $Q' = \langle V, E, H \cup H' \rangle$ of M-*ISAT*$(\mathcal{I})$ is said to be *starting point definite wrt.* Q iff there exists a function $f : E \rightarrow \{=, \neq\}$ such that $H' = \{u^- f(e) v^- \mid \langle u, r, v \rangle \in E\}$. We denote this relation by $def^-(Q, Q')$. This means that for each relation, either the starting points of related intervals are forced to be equal in all models, or they are forced to be distinct in all models. If for some Q we have $def^-(Q, Q')$, then Q' is said to be *starting point definite*.

Similarly, by exchanging starting and ending points, we get that Q' is *ending point definite wrt.* Q, denoted $def^+(Q, Q')$. □

Line 13 of Algorithm 3.10 checks the following, as we shall see.

Definition 3.12 (Locally satisfiable for starting (ending) points) Let $Q = \langle V, E, H\rangle$ be a starting point definite instance of *M-ISAT*$(\mathcal{I})$ for some $\mathcal{I} \subseteq \mathcal{A}$, and construct $Q' = \langle V, E', H\rangle$ such that

$$E' = \{\langle u, r, v\rangle \in E \mid u^- = v^- \in H\},$$

i.e. by considering only relations which force the starting points to be equal. Now Q is said to be *locally satisfiable for starting points* iff Q' is satisfiable. A model satisfying Q' is said to *locally satisfy* Q *for starting points.*

Similarly, exchanging starting and ending points, Q is said to be *locally satisfiable for ending points* iff Q' is satisfiable, and a model satisfying Q' is said to *locally satisfy* Q *for ending points.* $\square$

We now define the algebra for which Algorithm 3.10 solves satisfiability.

Definition 3.13 (Starting (ending) point algebra) A subalgebra $\mathcal{I} \subseteq \mathcal{A}$ is said to be a *starting point algebra* iff for any instance $Q = \langle V, E, H\rangle$ of *M-ISAT*$(\mathcal{I})$, the following holds: for any $T = \langle V, E, H'\rangle$ such that $\mathit{def}^-(\mathit{expl}^-(Q), T)$, if T is locally satisfiable for starting points, then T is satisfiable.

Symmetrically, exchanging ending points and starting points, we obtain an *ending point algebra.* $\square$

The satisfiability problems for these algebras are defined as follows.

Definition 3.14 (M_s-*ISAT*$(\mathcal{I})$, M_e-*ISAT*$(\mathcal{I})$) Let $\mathcal{I}$ be a starting point algebra. The *satisfiability problem for starting point algebras with metric information* is the set of instances $\langle V, E, H\rangle$ of *M-ISAT*$(\mathcal{I})$ where the DLRs of H are restricted in two ways: first, H may only contain Horn DLRs and second, H may not contain any variables v^+, where $v \in V$, i.e., it may only relate starting points of intervals.

Symmetrically, by exchanging starting and ending points, we get the *satisfiability problem for ending point algebras with metric information.* $\square$

4 Tractable Algebras

We now present the algebras which are starting (ending) point algebras.

Definition 4.1 (The subclasses $S(b)$ and $E(b)$) Set $r_s = (\succ\ \mathsf{d}\, \mathsf{o}^{\smile}\, \mathsf{m}^{\smile}\, \mathsf{f})$, and $r_e = (\prec\ \mathsf{d}\ \mathsf{o}\ \mathsf{m}\ \mathsf{s})$. Note that r_s contains all basic relations b such that whenever IbJ for interval variables I, J, $I^- > J^-$ has to hold in any model and symmetrically, r_e is equivalent to $I^+ < J^+$ holding in any model.

First, for $b \in \{\succ, \mathsf{d}, \mathsf{o}^{\smile}\}$, define $S(b)$ to be the set of relations r, such that either of the following holds:

$$\begin{array}{rcccl} (b\ b^{\smile}) & \subseteq & r & & \\ (b) & \subseteq & r & \subseteq & r_s \cup (\equiv\ \mathsf{s}\ \mathsf{s}^{\smile}) \\ (b^{\smile}) & \subseteq & r & \subseteq & {r_s}^{\smile} \cup (\equiv\ \mathsf{s}\ \mathsf{s}^{\smile}) \\ & & r & \subseteq & (\equiv\ \mathsf{s}\ \mathsf{s}^{\smile}). \end{array}$$

Then, by switching starting and ending points of intervals, for $b \in \{\prec, \mathsf{d}, \mathsf{o}\}$, $E(b)$ is defined to be the set of relations r, such that either of the following holds:

$$\begin{array}{rcccl} (b\ b^{\smile}) & \subseteq & r & & \\ (b) & \subseteq & r & \subseteq & r_e \cup (\equiv\ \mathsf{f}\ \mathsf{f}^{\smile}) \\ (b^{\smile}) & \subseteq & r & \subseteq & {r_e}^{\smile} \cup (\equiv\ \mathsf{f}\ \mathsf{f}^{\smile}) \\ & & r & \subseteq & (\equiv\ \mathsf{f}\ \mathsf{f}^{\smile}). \end{array}$$

□

Definition 4.2 (The subclasses S^* and E^*) Let r_s and r_e be as in Definition 4.1, and define S^* to be the set of relations r, such that either of the following holds:

$$\begin{array}{rcccl} (\equiv\ \mathsf{f}\ \mathsf{f}^{\smile}) & \subseteq & r & & \\ (\mathsf{f}\ \mathsf{f}^{\smile}) & \subseteq & r & \subseteq & r_s \cup {r_s}^{\smile} \\ (\equiv\ \mathsf{f}) & \subseteq & r & \subseteq & r_s \cup (\equiv\ \mathsf{s}\ \mathsf{s}^{\smile}) \\ (\equiv\ \mathsf{f}^{\smile}) & \subseteq & r & \subseteq & {r_s}^{\smile} \cup (\equiv\ \mathsf{s}\ \mathsf{s}^{\smile}) \\ (\mathsf{f}) & \subseteq & r & \subseteq & r_s \\ (\mathsf{f}^{\smile}) & \subseteq & r & \subseteq & {r_s}^{\smile} \\ (\equiv) & \subseteq & r & \subseteq & (\equiv\ \mathsf{s}\ \mathsf{s}^{\smile}) \\ & & r & = & \perp \end{array}$$

Symmetrically, replacing f by s (and their inverses), $(\equiv\ \mathsf{s}\ \mathsf{s}^{\smile})$ by $(\equiv\ \mathsf{f}\ \mathsf{f}^{\smile})$, and r_s by r_e, we get the subclass E^*. □

Files containing the algebras are supplied as an on-line appendix to this article.

It is easy to see that the algebras are all considerably larger than, for example, the ORD-Horn algebra, which contains 868 elements.

Proposition 4.3 The six algebras $S(b)$ and $E(b)$ contain 2312 elements each and S^* and E^* contain 1445 elements each.
Proof: A straightforward combinatorial exercise according to the definitions. $\square$

We also see that the $S(b)$ and $E(b)$ algebras each contain five basic relations, and that S^* and E^* contain three basic relations each. A subsumption result and a nonsubsumption result follow.

Proposition 4.4 The twelve algebras presented by Drakengren and Jonsson [1996], which were not classified as maximal tractable, are each included in one of the algebras $S(b)$ and $E(b)$.
Proof: By simply checking inclusion from the definitions. $\square$

Proposition 4.5 In all of the algebras $S(b)$, $E(b)$, S^* and E^*, there are relations which are not expressible by Horn DLRs alone.
Proof: It is easily verified that the point relations induced by the Allen relations $(\prec\ \succ)$, $(\mathsf{d}\ \mathsf{d}^{\smile})$, $(\mathsf{o}\ \mathsf{o}^{\smile})$, $(\succ\ \mathsf{f}^{\smile})$ and $(\prec\ \mathsf{s})$ are not Horn DLRs. $\square$

It was observed by Drakengren and Jonsson [1996] that the ORD-Horn algebra cannot express the notion of *sequentiality*, and thus since it is maximal tractable, we cannot add the relation $(\prec\ \succ)$ to it without losing tractability. However, we can obtain a weaker yet useful result by the following observation: We know from the results of Jonsson and Bäckström [1996] that the expressivity of Horn DLRs subsumes that of the ORD-Horn algebra, by expressing the ORD-Horn relations as disjunctions of point relations in the starting and ending points of the intervals. Thus, since the satisfiability problem for starting point algebras (and ending point algebras, which follows by symmetry) allow arbitrary Horn DLRs relating starting points, we can convert any network expressed in the ORD-Horn algebra into an equivalent instance of M_s-$ISAT(\mathcal{I})$ for some of the tractable subclasses above, where only starting points of intervals are related. The additional expressivity of the starting point algebras can then be used to express e.g. sequentiality (using one of the algebras $S(\succ)$ or $E(\prec)$) or other relations between intervals.

Now it is time to verify that the presented algebras are indeed starting and ending point algebras, respectively. A few auxiliary definitions and results are needed.

Definition 4.6 (Sign function) For $x \in \mathbb{R}$, let $sgn(x) \in \{-1, 0, 1\}$ be the *sign* of x, that is, if $x < 0$, then $sgn(x) = -1$, if $x = 0$ then $sgn(x) = 0$, and if $x > 0$, then $sgn(x) = 1$. $\square$

Lemma 4.7 Let $Q = \langle V, E, H \rangle$ be a starting point definite instance of M_s-$ISAT(\mathcal{I})$, which is locally satisfiable for starting points by some model M, and let M' be an interpretation for Q such that

- $\forall v \in V.M(v^-) = M'(v^-)$
- $\forall u, v \in V.M(u^-) = M(v^-) \rightarrow$
  $sgn(M(u^+) - M(v^+)) = sgn(M'(u^+) - M'(v^+))$,

which means that M' may not differ from M in the interpretation of starting points, and for ending points, any change is allowed, as long as their relative order for relations which have the same interpretations of starting points is the same. Then M' also locally satisfies Q for starting points.
Proof: Apart from checking the DLRs in H, local satisfiability for starting points checks only relations where the intervals they relate are forced to have the same starting point. Since H does not relate ending points of intervals, the only thing that affects satisfiability of these relations is the relative order of ending points, given a fixed starting point. Since this order is the same, and M and M' coincide on starting points, the result follows. □

Lemma 4.8 Let $Q = \langle V, E, H \rangle$ be a starting point definite instance of M_s-$ISAT(\mathcal{I})$, where for no $\langle u, r, v \rangle \in E$, $r \cap (\equiv \ \mathsf{s}\, \mathsf{s}^{\smile}) \neq \emptyset$ and $r - (\equiv \ \mathsf{s}\, \mathsf{s}^{\smile}) \neq \emptyset$. Then Q is satisfied by the model M iff M locally satisfies Q for starting points and M satisfies $\langle V, E' \rangle$, for $E' = \{\langle u, r, v \rangle \in E \mid r \cap (\equiv \ \mathsf{s}\, \mathsf{s}^{\smile}) = \emptyset\}$.
Proof:
$\Rightarrow$) Assuming that M satisfies Q, the latter condition is a direct consequence of the definitions.
$\Leftarrow$) By the restriction on Q, satisfiability of every relation r is checked by the two conditions together, and the satisfiability of H is included in the local satisfiability condition. Thus Q is satisfiable. □

Lemma 4.9 Let $Q = \langle V, E, H \rangle$ be an instance of M-$ISAT(\mathcal{I})$, and let $T = \langle V, E, H' \rangle$ be such that $def^-(Q, T)$. Construct $T' = \langle V, E', H' \rangle$ by setting

$$\begin{aligned} E' \quad = \quad & \{\langle u, r \cap (\equiv \ \mathsf{s}\, \mathsf{s}^{\smile}), v \rangle \mid v_1^- = v_2^- \in H\} \cup \\ & \{\langle u, r - (\equiv \ \mathsf{s}\, \mathsf{s}^{\smile}), v \rangle \mid v_1^- \neq v_2^- \in H\} \end{aligned}$$

Now T' is satisfiable iff T is. The analogous result holds for ending points, when references to starting points are changed to ending points, and $(\equiv \ \mathsf{s}\, \mathsf{s}^{\smile})$ is changed to $(\equiv \ \mathsf{f}\, \mathsf{f}^{\smile})$.

Proof: Directly from the definitions. The restrictions imposed on the qualitative relations are already guaranteed to hold in any model, by the restrictions on H'. Note that E' is well-defined since *def* adds to H either equality or inequality for all interval starting points.

The property of ending points follows by symmetry. □

Definition 4.10 (Absolute value) For $x \in \mathbb{R}$, denote by $abs(x)$ the absolute value of x, i.e. $x \cdot sgn(x)$. □

The main results follow.

Theorem 4.11 The algebras $S(b)$ are starting point algebras, and the algebras $E(b)$ are ending point algebras.

Proof: Let $Q = \langle V, E, H \rangle$ be an instance of M_s-$ISAT(S(b))$, and let $T = \langle V, E, H' \rangle$ with $def^-(expl^-(Q), T)$. By Lemma 4.9, we can assume that for every $\langle u, r, v \rangle \in E$, either $r \subseteq (\equiv\ \mathsf{s}\ \mathsf{s}^\smile)$ or $r \cap (\equiv\ \mathsf{s}\ \mathsf{s}^\smile) = \emptyset$, since T is starting point definite. Thus, the only relations r which are left are those satisfying

$$\begin{array}{rcccl} (b\ b^\smile) & \subseteq & r & \subseteq & r_s \cup r_s{}^\smile \\ (b) & \subseteq & r & \subseteq & r_s \\ (b^\smile) & \subseteq & r & \subseteq & r_s{}^\smile \\ & & r & \subseteq & (\equiv\ \mathsf{s}\ \mathsf{s}^\smile). \end{array}$$

So, suppose that T is locally satisfied for starting points by a model M. Since T has explicit starting points, all relations from line 4 are satisfied in M. Also, since M imposes a certain order on starting points of intervals, we know that without loss of generality, relations r on line 1 can be replaced by either $r \cap r_s$ or $r \cap r_s{}^\smile$, since r_s and $r_s{}^\smile$ impose disjoint orderings on starting points. Thus, without loss of generality, we take T only to contain relations from lines 2 and 4, inverting relations containing line 3 relations while changing their direction. If we could modify M into an interpretation M' such that the conditions of Lemma 4.7 are satisfied, and the relations of line 2 and their inverses are satisfied, then by Lemma 4.8, since these relations do not overlap with $(\equiv\ \mathsf{s}\ \mathsf{s}^\smile)$, M' would be a model of T, and the result would follow. Indeed, we shall satisfy line 2 relations with the basic relation b on every arc.

A few auxiliary definitions are needed for the construction. Define $MD(M)$ to be the least nonzero member of the set $\{abs(M(u^-) - M(v^-)) \mid u, v \in V\}$, i.e., the least nonzero distance between starting points in M. Given a $w \in V$, let $EP(w)$ be the set

$$\{v \in V \mid M(v^-) = M(w^-)\},$$

i.e., the set of intervals whose order of ending points has to be fixed, to maintain local satisfiability, by Lemma 4.7. Note that $EP(w)$ is uniquely determined by $M(w^-)$. Also, for a given w, let $n_w = |\{M(v^+) \mid v \in EP(w)\}|$ (the number of distinct ending points in M within w's "group"), and $f_w : EP(w) \to \{k \in \mathbb{N} \mid k < n_w\}$ the function uniquely determined by the ordering on ending points of $EP(w)$ in M, such that for every $u, v \in EP(w)$,

$$sgn(M(u^+) - M(v^+)) = sgn(f_w(u) - f_w(v)).$$

Note that for every $u, v \in EP(w)$, $f_u = f_v$.

Let $s = |\{M(v^-) \mid v \in V\}|$, i.e., the number of distinct starting points in M. Corresponding to f_w, define $i : V \to \{k \in \mathbb{N} \mid k < s\}$ to be the uniquely determined function such that for every $u, v \in V$,

$$sgn(M(u^-) - M(v^-)) = sgn(i(u) - i(w)).$$

We construct the interpretation M' as follows, depending on b, and afterwards prove that it is a model of T. First, set $\epsilon = MD(M)$, and for all $v \in V$, set $M'(v^-) = M(v^-)$.

- Suppose that b is $\succ$. For every $v \in V$, set

$$M'(v^+) = M(v^-) + \frac{\epsilon}{4}(1 + \frac{f_v(v)}{n_v}).$$

- Suppose that b is d. For every $v \in V$, set

$$M'(v^+) = M(v^-) + 1 + 2\epsilon(s - i(v) - 1) + \frac{\epsilon}{2}(\frac{f_v(v)}{n_v} - 1).$$

- Suppose that b is $\mathsf{o}^{\smile}$. First let $w \in V$ be such that $i(w) = s - 1$, i.e., having the largest ending point, set $M'(w^+) = M(w^-) + 1$. For every $v \in V$, set

$$M'(v^+) = M(w^-) + \frac{i(v) + 1}{s} + \frac{1}{s}(\frac{f_v(v)}{n_v} - \frac{1}{2}).$$

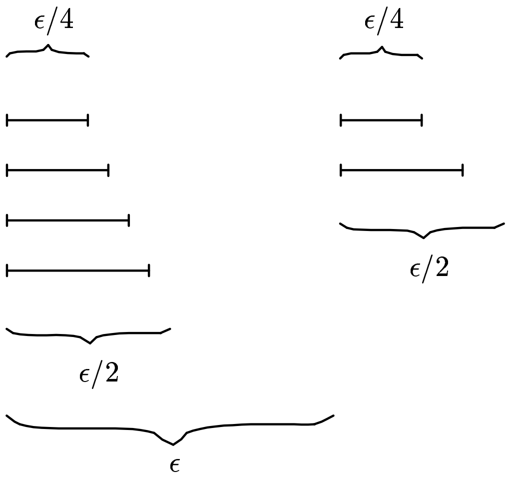

Figure 1: An example of the construction for the case when b is $\succ$ in the proof of Theorem 4.11.

Now, by construction, the models satisfy b on every arc. Furthermore, the order of ending points within each $EP(v)$ is retained, and the orderings on starting points are identical in M and M'. Thus M' locally satisfies T for starting points by Lemma 4.7, and by Lemma 4.8, M' is a model of T.

The proof for $E(b)$ is symmetrical. □

Fortunately, the next proof is much more convenient.

Theorem 4.12 The algebra S^* is a starting point algebra, and E^* is an ending point algebra.

Proof: We prove only the S^* case, since the E^* case is symmetrical.

Precisely as in the proof of Theorem 4.11, we obtain an instance T labelled only by relations r satisfying

$$\begin{array}{rcccl} (\mathsf{f}\ \mathsf{f}^{\smile}) & \subseteq & r & \subseteq & r_s \cup r_s{}^{\smile} \\ (\mathsf{f}) & \subseteq & r & \subseteq & r_s \\ (\mathsf{f}^{\smile}) & \subseteq & r & \subseteq & r_s{}^{\smile} \\ (\equiv) & \subseteq & r & \subseteq & (\equiv\ \mathsf{s}\ \mathsf{s}^{\smile}) \\ & & r & = & \bot. \end{array}$$

First assume that T is locally satisfied for starting points by some model M. Since M imposes a certain order on starting points of intervals, we know that relations r on line 1 now can be assumed to be either $r \cap r_s$ or $r \cap r_s{}^{\smile}$, since relations in r_s and $r_s{}^{\smile}$ impose disjoint orderings on starting points, as in the proof of Theorem 4.11. Thus, without loss of generality, we take T to contain only relations from lines 2 and 4, inverting line 3 relations while changing their direction (relations from line 5 make every instance unsatisfiable). We shall show that T is always satisfiable by constructing a model M' of T. First set $M'(v^-) = M(v^-)$ for every $v \in V$. Let x be maximal such that $x = M(v^-)$, i.e., x is the largest starting point in M. Then set $M'(v^+) = x + 1$ for every $v \in V$. Obviously, for $u, v \in V$, whenever $M'(u^-) > M'(v^-)$, then $u(\mathsf{f})v$ is satisfied in M'. For line 4 relations r, when urv, it has to hold that $M'(u^-) = M'(v^-)$, and thus by construction $M'(u^+) = M'(v^+)$, and r is satisfied. Thus M' is a model of T and the result follows. $\Box$

It remains to show that Algorithm 3.10 correctly solves satisfiability for starting and ending point algebras.

5 Correctness of the Algorithm

Several concepts are needed for the correctness proof.

Definition 5.1 (Convex combination, Convex) Given $x, y \in \mathbb{R}^n$, a *convex combination* of x and y is any $z \in \mathbb{R}^n$ of the form $\theta x + (1-\theta)y$, where $0 \leq \theta \leq 1$.

A set $S \subseteq \mathbb{R}^n$ is said to be *convex* iff any convex combination of elements in S is already in S. $\Box$

Definition 5.2 (Hyperplane) A *hyperplane* in $\mathbb{R}^n$ is a non-empty set defined as

$$\{x \in \mathbb{R}^n \mid \alpha(x) = b\},$$

for a linear polynomial α in $x_1, \ldots, x_n$, and $b \in \mathbb{R}$. $\Box$

Definition 5.3 (Almost convex) A set $S \subseteq \mathbb{R}^n$ is said to be *almost convex* if for any $x, y \in S$, at most finitely many of the convex combinations of x and y are not in S. $\Box$

Example 5.4 The set

$$S = \mathbb{R}^3 - \{(x, y, z) \mid x, y, z \in \mathbb{Z}\}$$

is almost convex. $\Box$

Proposition 5.5 Any intersection of finitely many almost convex sets is almost convex.

Proof: Let $S_1, \ldots, S_k$ be almost convex sets and let $S = S_1 \cap \ldots \cap S_k$. Take $x, y \in S$. Then $x, y \in S_i$ for every $1 \leq i \leq k$. Thus finitely many of the convex combinations of x and y are not in S_i. But since we intersect only a finite number of sets, only finitely many convex combinations are excluded from S, and the result follows. $\Box$

The following is a generalisation of Lemma 13 of Jonsson and Bäckström [1996] (which is a simplified version of a proof of the same result by Lassez and McAloon, 1992), from convex to almost convex sets.

Lemma 5.6 Let $S \subseteq \mathbb{R}^n$ be an almost convex set, and let $H_1, \ldots, H_k$ be hyperplanes. If $S \subseteq \bigcup_{i=1}^k H_i$, then there exists a j, $1 \leq j \leq k$ such that $S \subseteq H_j$.

Proof: Induction on k, letting $\mathcal{H}_k = \bigcup_{i=1}^k H_i$. For $k = 1$, the result is immediate, since $\mathcal{H}_1 = H_1$. Assuming that the result holds for k, we wish to show that if $S \subseteq \mathcal{H}_{k+1}$, then there exists a j, $1 \leq j \leq k+1$ such that $S \subseteq H_j$. So, suppose $S \subseteq \mathcal{H}_{k+1}$, i.e., that $S \subseteq \mathcal{H}_k \cup H_{k+1}$. If $S \subseteq \mathcal{H}_k$, then the result follows by induction. Also if $S \subseteq H_{k+1}$, or if one of $\mathcal{H}_k$ or H_{k+1} is included in the other, the result follows. Thus, suppose that $x, y \in S$, $x \in \mathcal{H}_k$, $y \in H_{k+1}$, satisfying $x \notin H_{k+1}$ and $y \notin \mathcal{H}_k$, which is the remaining case.

Consider the line segment L adjoining x and y, which is the convex combinations of x and y. Every hyperplane either contains L or intersects L in at most one point. If some H_i contains L, then $x, y \in H_i$, violating the choice of x and y. Thus L is intersected by hyperplanes H_i in at most finitely many points, and those are the only members of L which can be in $\mathcal{H}_{k+1}$. Since S is almost convex, there are infinitely many points of L which are in S and are not members of any hyperplane H_i, contradicting that this remaining case could hold. The result follows. $\Box$

One may note that this result holds even if S satisfies the weaker property of containing infinitely many convex combinations of each pair of elements, but then the proof of Proposition 5.5 would not go through.

Lemma 5.7 Let γ be a satisfiable Horn DLR, and let

$$S(\gamma) = \{x \in \mathbb{R}^n \mid \gamma \text{ satisfied by } x\}$$

be the set of all solutions to γ. Then $S(\gamma)$ is almost convex.

Proof: By the definition of a Horn DLR, for some convex set C and hyperplanes H_i, we have

$$S(\gamma) = C \cup \bigcup_{i=1}^{k} \overline{H_i} = C \cup \overline{\bigcap_{i=1}^{k} H_i}.$$

Take $x, y \in S(\gamma)$. If $x, y \in C$, then every convex combination of x and y is in $S(\gamma)$ by the convexity of C. If $y \notin C$ (and x is either in C or not), then $y \in \overline{H_l}$ for some l. Let L be the line segment adjoining x and y. The hyperplane H_l either contains L or intersects L in at most one point. It cannot contain L, since $y \notin H_l$; thus H_l intersects L in at most one point, and the remaining points of L are members of $\overline{H_l}$, and thus are members of $S(\gamma)$. □

Lemma 5.8 Let H be a satisfiable set of Horn DLRs, and let

$$S(H) = \{x \in \mathbb{R}^n \mid \text{every } \gamma \in H \text{ is satisfied by } x\}$$

be the set of all solutions to H. Then $S(H)$ is almost convex.
Proof: Directly from Lemma 5.7 and Proposition 5.5. □

The following is the key result for obtaining correctness of the algorithm.

Theorem 5.9 Let H be a satisfiable set of Horn DLRs, and let $x_1, x_2, \ldots, x_n$ be the variables used in H. Also define the set Γ of Horn DLRs by

$$\Gamma = \{x_i \neq x_j \mid \{x_i \neq x_j\} \cup H \text{ is satisfiable}\}.$$

Then $H \cup \Gamma$ is satisfiable.
Proof: Suppose that $H \cup \Gamma$ is not satisfiable, although H is satisfiable. Let S be the set of solutions of H, as defined in Lemma 5.8. By definition, the set of solutions to a formula $x_i \neq x_j \in \Gamma$ is a complement of some hyperplane H_{ij}. Thus, the set of solutions to $H \cup \Gamma$ is $S - \bigcup_{i,j} H_{ij} = \emptyset$, equivalent to $S \subseteq \bigcup_{i,j} H_{i,j}$. By Lemma 5.8, S is almost convex, and by Lemma 5.6, $S \subseteq H_{kl}$ for some hyperplane H_{kl}. But then $S \cap H_{kl} = \emptyset$, contrary to our assumption, and the result follows. □

Thus, it is enough to check the formulae in Γ *separately* for satisfiability. The correctness proof of the algorithm follows.

Theorem 5.10 Algorithm 3.10 correctly solves satisfiability for starting point algebras. Symmetrically, by exchanging starting and ending points of Algorithm 3.10, it correctly solves satisfiability for ending point algebras.

Proof: Suppose that Q is satisfiable. Then after line 1, Q' is satisfiable, by Proposition 3.9. Thus we cannot get reject at line 3, since H' has to be satisfied for Q' to be satisfiable. Consider the value of K at line 12, and set $K' = \{u^- \neq v^- \mid u^- \neq v^- \in K\}$. By the construction of K in lines $4 - 11$, all models of Q' have to satisfy the formulae of $K - K'$. Further, using Theorem 5.9 and the construction of K, $H' \cup K'$ is also satisfiable, and thus $H' \cup K$ is satisfiable. Now, by the construction of K, line 13 only tests relations which have to hold in any model of $\langle V, E, H' \cup K\rangle$, and thus cannot reject, since then Q' would not be satisfiable. Consequently, the algorithm accepts.

Suppose that Q is not satisfiable, and that it accepts, meaning that neither of the tests at lines 2 or 13 succeed. At line 2, Q' is not satisfiable, by Proposition 3.9. At line 4, H' is satisfiable, and by the construction of K in lines $4 - 11$, we have $\mathit{def}^+(Q', \langle V, E, H' \cup K\rangle)$ at line 12. By the same argument as above, $H' \cup K$ is satisfiable at line 12, using Theorem 5.9. If $\langle V, E, H' \cup K\rangle$ were locally satisfiable for starting points, then $\langle V, E, H' \cup K\rangle$ would be satisfiable, since $\mathcal{I}$ is a starting point algebra, and thus Q' would be satisfiable, contrary to our assumption. Thus $\langle V, E, H' \cup K\rangle$ is not locally satisfiable.

Since the algorithm does not reject in line 14, the set

$$P = \{u^+ \mathit{eprel}^-(r) v^+ \mid \langle u, r, v\rangle \in E \wedge u^- = v^- \in H' \cup K\}$$

is satisfiable by some model M. We already know that there exists a model N for $H' \cup K$. We shall construct an interpretation M' from M and N which locally satisfies $\langle V, E, H' \cup K\rangle$, a contradiction.

Let x be minimal such that for some $v \in V$, $x = M(v^+)$, i.e. the smallest ending point in M, and let y be maximal such that for some $v \in V$, $y = N(v^-)$, i.e. the largest starting point in N. Now, for every $v \in V$, set $M'(v^-) = N(v^-)$ and $M'(v^+) = M(v^+) - x + y + 1$. We see that M' still satisfies P, since the order on ending points is identical in M and M', and that it satisfies $H' \cup K$, since M and M' coincide on starting points. Furthermore, setting

$$E' = \{\langle u, r, v\rangle \in E \mid u^- = v^- \in H' \cup K\},$$

$\langle V, E', H' \cup K\rangle$ is satisfied in M', since $H' \cup K$ and P are, and since by construction $M(v^-) < M(v^+)$ is satisfied for every $v \in V$. But this is the same as M' locally satisfying $\langle V, E, H' \cup K \rangle$, and this contradicts the fact that the algorithm does not reject. The result follows. $\Box$

Theorem 5.11 Algorithm 3.10 runs in polynomial time.
Proof: The transformation in line 1 is clearly polynomial and by Proposition 3.3, so is line 2. The loop between lines 5 to 11 is executed as many times as there are arcs in E, and lines 6 to 10 take only polynomial time, thus the loop takes polynomial time. The final test on line 12 is also polynomial-time (even linear), by Proposition 3.5, and the result follows. $\Box$

Corollary 5.12 M_s-*ISAT*$(\mathcal{I})$ for starting point algebras, and M_e-*ISAT*$(\mathcal{I})$ for ending point algebras, is solvable in polynomial time.
Proof: From Theorem 5.10 and Theorem 5.11. $\Box$

For instances $Q = \langle V, E, H \rangle$ where H only contains point algebra formulae, we can use PASAT instead of HORNDLRSAT in lines 2 and 6, since $expl^{-}(Q)$ adds only point algebra formulae to H and K in Algorithm 3.10 also contains only point algebra formulae. This allows for a much better worst-case complexity, as we shall see.

Corollary 5.13 If Algorithm 3.10 uses PASAT instead of HORNDLRSAT in lines 2 and 6, its complexity becomes in $O(|E|^2)$.
Proof: The first transformation clearly takes $O(|E|)$ time. The initial and final tests take $O(|E|)$ time, by Proposition 3.5, and the loop is executed $|E|$ times, each taking $O(|E|)$ time. Thus, the resulting complexity is $O(|E|^2)$. $\Box$

Note that we do not have a proof that these algebras are the *only* algebras which are starting or ending point algebras in Allen's interval algebra. It seems likely that there should be more algebras with the same structure. One advantage of the results presented in this paper is that once a starting or ending point algebra is found (and proved to be one), both a polynomial-time algorithm for satisfiability and the extension to metric temporal information are obtained for free.

6 Maximality of Tractable Subclasses

Recently, the search for tractable subalgebras of Allen's interval algebra has become more systematic, by focusing on finding tractable algebras that are *maximal*, in the sense that no algebra strictly containing it is tractable. The pioneering work by Nebel and Bürckert [1995] was to find a maximal tractable subclass containing all the thirteen basic relations, the *ORD-Horn* algebra, which in addition is the *unique* maximal algebra containing all the basic relations. The ORD-Horn algebra contains 868 relations. More recently, Drakengren and Jonsson [1996] have identified nine more maximal tractable algebras, eight of which are of size 2178, and one of size 4097. However, the latter is found to be of no use, since any instance of satisfiability for that algebra is always satisfiable, unless it contains the relation $\perp$. The former algebras each contain three basic relations.

In Proposition 4.4 we saw that the twelve nonmaximal algebras of Drakengren and Jonsson [1996] are included in the algebras of the current paper. However, it remains to check whether they are maximal or not.

One of the main tools for analysing maximal tractability is a *closure operation* on subclasses of the algebra, which preserves tractability.

Definition 6.1 (Closure) Let $S \subseteq \mathcal{A}$. Then we denote by $\overline{S}$ the *closure* of S under converse, intersection and composition, i.e., the least subalgebra (which is uniquely determined) containing S, which is closed under the three operations. $\square$

The key result for extrapolating tractability results is the following.

Proposition 6.2 Let $S \subseteq \mathcal{A}$. Then $\mathit{ISAT}(S)$ is polynomial-time iff $\mathit{ISAT}(\overline{S})$ is, and $\mathit{ISAT}(S)$ is NP-complete iff $\mathit{ISAT}(\overline{S})$ is.
Proof: See Nebel and Bürckert [1995]. $\square$

Our main tools for proving intractability are the following NP-complete subalgebras of $\mathcal{A}$.

Definition 6.3 (NP-complete algebras $\mathcal{N}_1$, $\mathcal{N}_2$ and Δ_0) First let

$$A = \{(\prec\ \mathsf{d}^\smile\ \mathsf{o}\ \mathsf{m}\ \mathsf{f}^\smile), (\prec\ \mathsf{d}\ \mathsf{o}\ \mathsf{m}\ \mathsf{s})\},$$

define

$$\mathcal{N}_1 = A \cup \{(\mathsf{d}\ \mathsf{d}^\smile\ \mathsf{o}^\smile\ \mathsf{s}^\smile\ \mathsf{f})\}$$

and

$$\mathcal{N}_2 = A \cup \{(\mathsf{d}^\smile\ \mathsf{o}\ \mathsf{o}^\smile\ \mathsf{s}^\smile\ \mathsf{f}^\smile)\}.$$

Also define

$$\Delta_0 = \{(\equiv\ \mathsf{d}\ \mathsf{d}^{\smile}\ \mathsf{o}\ \mathsf{o}^{\smile}\ \mathsf{m}\ \mathsf{m}^{\smile}\ \mathsf{s}\ \mathsf{s}^{\smile}\ \mathsf{f}\ \mathsf{f}^{\smile}), (\prec\ \succ)\}.$$

□

Proposition 6.4 $ISAT(\mathcal{N}_1)$, $ISAT(\mathcal{N}_2)$ and $ISAT(\Delta_0)$ are all NP-complete.

Proof: For $\mathcal{N}_1$ and $\mathcal{N}_2$, see Nebel and Bürckert [1995], and for Δ_0, see Golumbic and Shamir [1993]. □

Next, we prove maximality of the current algebras.

Proposition 6.5 The algebras $S(b)$, $E(b)$, S^*, E^* are maximal tractable.
Proof: By running the utility `atry` [Nebel and Bürckert, 1993], which generates minimal extensions of subclasses by adding a relation and computing the closure of that class. All extensions of these algebras contain either of the NP-complete algebras $\mathcal{N}_1$, $\mathcal{N}_2$ or Δ_0, so the result follows by Proposition 6.2.
□

Briefly, we show that the restriction that we cannot express starting and ending point information at the same time is essential for obtaining tractability, once we want to go outside the ORD-Horn algebra.

Proposition 6.6 Let $S \subseteq \mathcal{A}$ such that S is not a subset of the ORD-Horn algebra, and let SE be the set of instances $Q = \langle V, E, H\rangle$ of $M\text{-}ISAT(S)$, where H may contain only DLRs $u^+ = v^-$ for some $u, v \in V$. Then the satisfiability problem for SE is NP-complete.
Proof: By definition, S has to contain some relation outside the ORD-Horn algebra. Since the expression $u^+ = v^-$ is allowed in H, we can express the basic relation m, and assume that it is included in S. It is easily verified that $\overline{\{\mathsf{m}\}}$ contains all the basic relations (using e.g. the utility `aclose` [Nebel and Bürckert, 1993]), and thus $\overline{S}$ contains all basic relations. Since any subclass of Allen's algebra containing all the basic relations has to be contained in the ORD-Horn algebra in order to be tractable, and is NP-complete otherwise [Nebel and Bürckert, 1995], NP-completeness follows. □

Despite this result, it seems possible to express constraints on *duration* of intervals, without restricting their absolute position in time, and still obtain a polynomial-time algorithm, but this is left for future work.

Now, one might question the relevance of this work on the grounds that this might be just eight out of thousands (or more) of tractable subalgebras. That is, we would like to have a result similar to the unique maximality result of the ORD-Horn class, showing that these algebras are the only algebras satisfying some specific criterion of relevance. In fact, recent results by the authors [Drakengren and Jonsson, 1997] state that any tractable subclass that is not yet known in the literature cannot contain more than three basic relation including converses (technically, it cannot contain basic relations other than $(\equiv)$, (b) and $(b^{\smile})$ for $b \in \{\mathsf{d}, \mathsf{o}, \mathsf{s}, \mathsf{f}\}$). This means, for instance, that the two algebras $S(\succ)$ and $E(\prec)$ are the only maximal tractable algebras containing the relation $\prec$, and that any yet unpublished tractable subclass has to be less expressive in terms of the number of basic relations than the present algebras containing five basic relations.

7 Discussion

It seems appropriate to summarise the status of the search for maximal tractable subclasses of Allen's interval algebra: The eight new maximal tractable subalgebras presented in this paper increase the number of currently known maximal tractable subclasses to eighteen, including the ORD-Horn class [Nebel and Bürckert, 1995], and the nine algebras found by Drakengren and Jonsson [1996].

One may note that there is a considerable overlap between these, since the sizes of the algebras are 868 (one), 2178 (eight), 2312 (eight), and 4097 (one), the sum of the sizes being much more than 8192. For instance, we showed in Drakengren and Jonsson [1996] that the only relations of the ORD-Horn algebra that is not included in any of the algebras discussed in that paper are the relations (m) and $(\mathsf{m}^{\smile})$.

Of course, the ultimate goal is to classify the *complete* set of maximal tractable subalgebras, but since there are 2^{8192} subclasses to investigate, this is clearly a nontrivial task. Recent results on the *RCC-5 algebra* for spatial reasoning [Jonsson and Drakengren, 1997] show that brute-force methods can have success in characterising the complete set of tractable subclasses. The present problem is harder with several orders of magnitude, however, since the RCC-5 algebra contains only 2^{32} relations. Nevertheless, it is encouraging to note that there are only four maximal tractable subclasses of the RCC-5 algebra, out of the approximately $4.3 \cdot 10^9$ subclasses. As mentioned above, recent results by the authors [Drakengren and Jonsson, 1997] also provide a partial classification of tractability in Allen's algebra, using similar methods.

Concerning metric time, it still remains to provide the nine maximal tractable algebras of Drakengren and Jonsson [1996] with some kind of metric temporal information. A simple examination shows that we cannot use the present technique, since this would make the resulting algebras NP-complete (just add the qualitative relations induced by relations on starting or ending points, run `aclose` [Nebel and Bürckert, 1993], and verify that at least one of the NP-complete algebras of Proposition 6.4 is included), so for this, some other kind of expressivity is needed. Searching for other starting or ending point algebras could also be fruitful. Further, it seems possible that the techniques presented here can also be used for extending the point-interval algebra [Vilain, 1982] with metric time.

8 Conclusions

We have found eight new maximal tractable subclasses of Allen's interval algebra, and provided them with metric temporal information on starting or ending points of intervals, using the formalism of Horn DLRs [Jonsson and Bäckström, 1996]. Apart from representing progress in the research aiming at a complete characterisation of the tractable subclasses of Allen's interval algebra, this opens for a combination between the expressivity of the ORD-Horn algebra and algebras which can express *sequentiality* between intervals, provided that only starting or ending points of intervals are related with ORD-Horn relations.

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Paper IV

Towards a Complete Classification of Tractability in Allen's Algebra

Thomas Drakengren and Peter Jonsson

ABSTRACT

We characterise the set of subalgebras of Allen's algebra which have a tractable satisfiability problem, and in addition contain certain basic relations. The conclusion is that no tractable subalgebra that is not known in the literature can contain more than the three basic relations $(\equiv)$, (b) and $(b^{\smile})$, where $b \in \{\mathsf{d}, \mathsf{o}, \mathsf{s}, \mathsf{f}\}$. This means that concerning algebras for specifying *complete knowledge* about temporal information, there is no hope of finding yet unknown classes with much expressivity. Furthermore, we show that there are exactly two maximal tractable algebras which contain the relation $(\prec\ \succ)$. Both of these algebras can express the notion of *sequentiality*; thus we have a complete characterisation of tractable inference using that notion.

1 Introduction

This paper improves on known results about algorithms for the problem of reasoning about temporal constraints. Such reasoning is an important task in many areas of AI and elsewehere, such as planning [Allen, 1991], natural language processing [Song and Cohen, 1988], time serialization in archeology [Golumbic and Shamir, 1993] and more, and there are several frameworks for formalising such problems, according to different needs. Among the most frequently used ones are the *point algebra* [van Beek and Cohen, 1990], used for expressing qualitative relations between time points, the *point-interval* algebra [Vilain, 1982] for expressing qualitative relations between time points and time intervals, and the famous interval algebra of Allen [1983] for expressing qualitative relations between time intervals. There are also combinations of these and extensions to handle metric time as well, such as Meiri's framework [Meiri, 1991], and the works of Kautz and Ladkin [1991], Gerevini *et al.* [1993], Dechter *et al.* [1991], Jonsson and Bäckström [1996] and Drakengren and Jonsson [1997]. However, it was early proved that the reasoning problem for these formalisms is very hard; e.g. reasoning in Allen's interval algebra is NP-complete [Vilain and Kautz, 1986], and NP-hardness carries over to more expressive formalisms.

These computational problems have motivated the search for various tractable fragments of the temporal formalisms, where reasoning can be guaranteed to be reasonably efficient. In particular, several subclasses of Allen's algebra have been reported as tractable (we assume $P \neq NP$) [van Beek and Cohen, 1990; Golumbic and Shamir, 1993; Nebel and Bürckert, 1995; Drakengren and Jonsson, 1996; Drakengren and Jonsson, 1997]. However, in view of the large number of possible subclasses of Allen's algebra (the algebra contains 8192 relations, leading to $2^{8192} \approx 10^{2466}$ subclasses), such results are in danger of appearing *ad hoc*. As a first reaction to this, research has recently focused on identifying *maximal* tractable subclasses; i.e. classes which cannot be extended without losing tractability. This direction is clearly more systematic, since any tractable subclass is included in a maximal tractable one. The first such algebra was identified by Nebel and Bürckert [1995], soon to be followed by Drakengren and Jonsson [1996, 1997], resulting in eighteen known maximal algebras, subsuming all algebras previously known to be tractable. Still, however, this is a very small number compared to the total number of possible subclasses.

Due to this apparent lack of systematicity, techniques have recently been developed allowing *full* classifications of tractability, in particular for the point-interval algebra [Jonsson *et al.*, 1996], and also for the RCC-5 algebra for *spatial* reasoning [Jonsson and Drakengren, 1997]. A full classification of tractability for an algebra means that we identify the *complete* set of tractable subclasses in the algebra. Despite the success for the point-interval algebra and the RCC-5 algebra, the corresponding task for Allen's algebra poses a problem more difficult by several orders of magnitude: the number of subclasses in these algebras is only $2^{32} \approx 4.3 \cdot 10^9$. In principle, all these can be enumerated on a computer, but this is certainly not the case with the Allen algebra.

In this context, this paper presents a significant step towards a full classification of tractability in Allen's algebra. We show that any algebra that is yet to be found can contain at most three basic relations: $(\equiv)$, (b) and $(b^{\smile})$, for $b \in \{\mathsf{d}, \mathsf{o}, \mathsf{s}, \mathsf{f}\}$. This means that in order to specify complete temporal knowledge, we cannot hope to find more expressive algebras than those already known. Furthermore, we show that there are exactly two maximal tractable algebras which can express the important notion of *sequentiality* [Sandewall, 1994].

Finally, note that the main results of this paper are proved using exhaustive search by computers. Naturally, such proofs cannot be reproduced in a paper, but we encourage researchers in the field to repeat our proofs. All software mentioned in the paper can be obtained from the authors.

The structure of the paper follows. First we present Allen's algebra in Section 2, after which the classification results follow. A discussion concludes the paper. The more complicated proofs are collected in an appendix.

2 Allen's Algebra

Allen's interval algebra [Allen, 1983] is based on the notion of *relations between pairs of intervals*. An interval x is represented as a tuple $\langle x^-, x^+ \rangle$ of real numbers with $x^- < x^+$, denoting the left and right endpoints of the interval, respectively, and relations between intervals are composed as disjunctions of *basic interval relations*, which are those in Table 1 (we denote the set of these relations by $\mathbf{B}$). Such disjunctions are represented as *sets* of basic relations, but using a notation such that, for example, the disjunction of the basic intervals $\prec$, m and $\mathsf{f}^{\smile}$ is written $(\prec\ \mathsf{m}\ \mathsf{f}^{\smile})$. Thus, we have that $(\prec\ \mathsf{f}^{\smile}) \subseteq (\prec\ \mathsf{m}\ \mathsf{f}^{\smile})$. Sometimes, the disjunction of *all* basic relations is written $\top$, and the empty relation is written $\perp$ (this also used for relations between interval endpoints, denoting "always satisfiable" and

Basic relation		Example	Endpoints
x before y	$\prec$	xxx	$x^+ < y^-$
y after x	$\succ$	yyy	
x meets y	m	xxxx	$x^+ = y^-$
y met-by x	$\mathsf{m}^\smile$	yyyy	
x overlaps y	o	xxxx	$x^- < y^- < x^+$,
y overl.-by x	$\mathsf{o}^\smile$	yyyy	$x^+ < y^+$
x during y	d	xxx	$x^- > y^-$,
y includes x	$\mathsf{d}^\smile$	yyyyyyy	$x^+ < y^+$
x starts y	s	xxx	$x^- = y^-$,
y started by x	$\mathsf{s}^\smile$	yyyyyyy	$x^+ < y^+$
x finishes y	f	xxx	$x^+ = y^+$,
y finished by x	$\mathsf{f}^\smile$	yyyyyyy	$x^- > y^-$
x equals y	$\equiv$	xxxx	$x^- = y^-$,
		yyyy	$x^+ = y^+$

Table 1: The thirteen basic relations. The endpoint relations $x^- < x^+$ and $y^- < y^+$ that are valid for all relations have been omitted.

"unsatisfiable", respectively). The algebra is provided with the operations of *converse, intersection and composition* on intervals, but we shall need only the converse operation. The converse operation takes an interval relation i to its converse $i^\smile$, obtained by inverting each basic relation in i, i.e., exchanging x and y in the endpoint relations of Table 1.
By the fact that there are thirteen basic relations, we get $2^{13} = 8192$ possible relations between intervals in the full algebra. We denote the set of all interval relations by $\mathcal{A}$. Subclasses of the full algebra are obtained by considering subsets of $\mathcal{A}$. There are $2^{8192} \approx 10^{2466}$ such subclasses.

There are several problems of computation associated with Allen's interval algebra, and this paper focuses on the problem of *satisfiability* of a set of interval variables with relations between them, i.e. deciding whether there exists an assignment of intervals on the real line for the interval variables, such that all of the relations between the intervals are satisfied. We define this as follows.

Definition 2.1 ($\mathcal{A}$-SAT($\mathcal{I}$)) Let $\mathcal{I}$ be a set of interval relations. An instance of $\mathcal{A}$-SAT($\mathcal{I}$) is a labelled directed graph $S = \langle V, E \rangle$, where the nodes in V are interval variables and E is a subset of $V \times \mathcal{I} \times V$. A labelled edge $\langle u, r, v\rangle \in E$ means that u and v are related by r.

A function M taking an interval variable v to its interval representation $M(v) = \langle x^-, x^+\rangle$ with $x^- < x^+$, $x^-, x^+ \in \mathbb{R}$, is said to be an *interpretation* of S.

An instance $\langle V, E\rangle$ is said to be *satisfiable* iff there exists an interpretation M such that for each $\langle u, r, v\rangle \in E$, $M(u) r M(v)$ holds, i.e. the endpoint relations required by r (see Table 1) are satisfied by the assignments of u and v. Then M is said to be a *model* of $\langle V, E\rangle$.

We refer to the *size* of an instance $\langle V, E\rangle$ as $|V| + |E|$. □

For $\mathcal{A}$, we have the following result.

Proposition 2.2 $\mathcal{A}$-SAT($\mathcal{A}$) is NP-complete.
Proof: See Vilain and Kautz [1986]. □

Next, we introduce Nebel and Bürckert's [1995] *closure* operation, here denoted $\mathcal{C}_{\mathcal{A}}(\cdot)$, which transforms a given subclass of $\mathcal{A}$ to one that is polynomially equivalent to the original subclass wrt. satisfiability.

Definition 2.3 (Closure) Let $\mathcal{S} \subseteq \mathcal{A}$. Then we denote by $\mathcal{C}_{\mathcal{A}}(\mathcal{S})$ the $\mathcal{A}$-*closure* of $\mathcal{S}$, defined as the least subalgebra of $\mathcal{A}$ containing $\mathcal{S}$ and which is closed under converse, intersection and composition. □

Closures can be computed using Nebel and Bürckert's software [1993].

The key result for extrapolating complexity results is the following.

Proposition 2.4 For $\mathcal{S} \subseteq \mathcal{A}$, $\mathcal{A}$-SAT($\mathcal{S}$) is polynomial iff $\mathcal{A}$-SAT($\mathcal{C}_{\mathcal{A}}(\mathcal{S})$) is, and $\mathcal{A}$-SAT($\mathcal{S}$) is NP-complete iff $\mathcal{A}$-SAT($\mathcal{C}_{\mathcal{A}}(\mathcal{S})$) is.
Proof: See Nebel and Bürckert [1995]. □

$\mathcal{A}$-SAT is sometimes defined such that for each pair of objects (e.g. time intervals), we have exactly one relation (cf. Golumbic and Shamir [1993]). In this way, the reduction needed for Proposition 2.4 would fail, since intervals which are added are not always related.

3 Classification of $\mathcal{A}$

This section contains the parts of the classification.

3.1 Intractable Subclasses

In order to provide the classification, we need to find more NP-complete subclasses of $\mathcal{A}$ than those previously known. Our main tools for proving intractability are the following NP-complete subclasses of $\mathcal{A}$.

Definition 3.1 (Subclasses $\mathcal{N}_i$, relation R, sets $\mathcal{A}_0$, $\mathcal{A}_4$ and $\mathcal{A}_{NP}$) First define the auxiliary set A by $A = \{(\prec\ \mathsf{d}^{\smile}\ \mathsf{o}\ \mathsf{m}\ \mathsf{f}^{\smile}), (\prec\ \mathsf{d}\ \mathsf{o}\ \mathsf{m}\ \mathsf{s})\}$. Define the following sets.

$$\mathcal{N}_1 = A \cup \{(\mathsf{d}\ \mathsf{d}^{\smile}\ \mathsf{o}^{\smile}\ \mathsf{s}^{\smile}\ \mathsf{f})\},$$

$$\mathcal{N}_2 = A \cup \{(\mathsf{d}^{\smile}\ \mathsf{o}\ \mathsf{o}^{\smile}\ \mathsf{s}^{\smile}\ \mathsf{f}^{\smile})\},$$

$$\mathcal{N}_3 = \{(\prec\ \succ), (\mathsf{o}\ \mathsf{o}^{\smile})\},$$

$$\mathcal{N}_4 = \{(\prec\ \succ), (\mathsf{o}\ \mathsf{o}^{\smile}\ \mathsf{m}\ \mathsf{m}^{\smile})\},$$

$$\mathcal{N}_5 = \{(\mathsf{m}\ \mathsf{m}^{\smile}), (\prec\ \succ\ \mathsf{s}\ \mathsf{s}^{\smile}\ \mathsf{f}\ \mathsf{f}^{\smile})\}.$$

Define the relations R and R' by

$$R = (\mathsf{d}\ \mathsf{d}^{\smile}\ \mathsf{o}\ \mathsf{o}^{\smile})$$

and

$$R' = (\equiv\ \mathsf{m}\ \mathsf{m}^{\smile}\ \mathsf{s}\ \mathsf{s}^{\smile}\ \mathsf{f}\ \mathsf{f}^{\smile}),$$

and set $\mathcal{A}_{NP}$ to be the union of the following sets:

$$\{\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4, \mathcal{N}_5\},$$

$$\mathcal{A}_0 = \{\{(\prec\ \succ), r\} \mid R \subseteq r \subseteq R \cup R'\},$$

$$\mathcal{A}_4 = \{\{(\prec\ \succ), r\} \mid R \cup (\prec) \subseteq r \subseteq R \cup (\prec) \cup R'\}.$$

□

Proposition 3.2 $\mathcal{A}$-SAT($\mathcal{S}$) is NP-complete for all $\mathcal{S} \in \mathcal{A}_{NP}$.
Proof: For $\mathcal{N}_1$ and $\mathcal{N}_2$, see Nebel and Bürckert [1995]. The remaining cases are proved in Theorem A.9 ($\mathcal{A}_0 \cup \mathcal{A}_4$) and Theorem A.12 ($\mathcal{N}_3$, $\mathcal{N}_4$ and $\mathcal{N}_5$) in the appendix. □

3.2 Tractable Algebras

Next we define what are the polynomial algebras involved in the classification.

Definition 3.3 ($bas(A)$, polynomial algebras) Let $bas(A)$ for $A \subseteq \mathcal{A}$ be the set of basic relations contained in A. Also let $\mathcal{H}$ denote the ORD-Horn algebra by Nebel and Bürckert [1995] and $\mathcal{S}_{\prec}$, $\mathcal{S}_{\mathsf{d}}$, $\mathcal{S}_{\mathsf{o}}$, $\mathcal{E}_{\prec}$, $\mathcal{E}_{\mathsf{d}}$ and $\mathcal{E}_{\mathsf{o}}$ the maximal tractable algebras of Drakengren and Jonsson [1997], where $\mathcal{S}_r$ denotes the unique *starting point algebra* containing the basic relation (r), and $\mathcal{E}_r$ the unique *ending point algebra* containing the basic relation (r). □

The following facts about the algebras will be needed in the classification.

Proposition 3.4 $\mathcal{H}$, $\mathcal{S}_r$ and $\mathcal{E}_r$ are maximal tractable subclasses of $\mathcal{A}$, i.e. it is impossible to extend them without losing tractability. Furthermore, any tractable subclass $A \subseteq \mathcal{A}$ with $\mathbf{B} \subseteq A$ satisfies $A \subseteq \mathcal{H}$. Also, $bas(\mathcal{H}) = \mathbf{B}$,

$$bas(\mathcal{S}_r) = \{\equiv, r, r^{\smile}, \mathsf{s}, \mathsf{s}^{\smile}\},$$

and

$$bas(\mathcal{E}_r) = \{\equiv, r, r^{\smile}, \mathsf{f}, \mathsf{f}^{\smile}\}$$

for all $r \in \{\prec, \mathsf{d}, \mathsf{o}\}$.

Proof: The proofs for $\mathcal{H}$ can be found in [Nebel and Bürckert, 1995], and those for $\mathcal{S}_r$ and $\mathcal{E}_r$ in [Drakengren and Jonsson, 1997]. □

In order to define the subject of our classification, define $\mathcal{T}$ to be the set of maximal tractable subalgebras of $\mathcal{A}$ not included in $\mathcal{H}$, $\mathcal{S}_r$ or $\mathcal{E}_r$, for any $r \in \{\prec, \mathsf{d}, \mathsf{o}\}$. Note that it is sufficient to restrict the attention to maximal tractable algebras, since any tractable subset can be extended to such an algebra. Also note that some of the algebras known from the literature (those of Drakengren and Jonsson [1996, 1997]) are included in $\mathcal{T}$, but this will not affect the classification, since these all contain three basic relations or less.

3.3 The Classification

We start by stating the main theorem of the paper, from which the classification results will follow. Since this kind of result has already been presented at least twice in the literature [Jonsson *et al.*, 1996; Jonsson and Drakengren, 1997], we take the opportunity to abstract it in order to make future classification results easier to state.

Theorem 3.5 Let $\mathcal{R}$ be a set equipped with an operation $\mathcal{C}_{\mathcal{R}}(R)$ on sets $R \subseteq \mathcal{R}$, and for each set $R \subseteq \mathcal{R}$ a problem $\mathcal{R}$-SAT(R), satisfying the following:

- If $\mathcal{R}$-SAT$(\mathcal{C}_{\mathcal{R}}(R))$ is NP-complete, then $\mathcal{R}$-SAT(R) is NP-complete
- If $\mathcal{R}$-SAT(R) is NP-complete, then $\mathcal{R}$-SAT(S) is NP-complete for all $S \supseteq R$
- If $\mathcal{R}$-SAT(R) is polynomial, then $\mathcal{R}$-SAT(S) is polynomial for all $S \subseteq R$.

Let $\mathcal{R}_P, \mathcal{R}_{NP} \subseteq 2^{\mathcal{R}}$ and $\mathcal{B} \subseteq \mathcal{R}$, such that $\mathcal{R}$-SAT(X) is polynomial for each $X \in \mathcal{R}_P$, each $X \in \mathcal{R}_P$ satisfies $\mathcal{B} \subseteq X$, and $\mathcal{R}$-SAT(D) is NP-complete for each $D \in \mathcal{R}_{NP}$.

Then if each set $T \subseteq \mathcal{R}$ with $|T| \leq |\mathcal{R}_P|$ satisfies either that T is a subset of some set in $\mathcal{R}_P$, or that $D \subseteq \mathcal{C}_{\mathcal{R}}(T \cup \mathcal{B})$ for some $D \in \mathcal{R}_{NP}$, then for any $\mathcal{S}$ with $\mathcal{B} \subseteq \mathcal{S}$, $\mathcal{R}$-SAT$(\mathcal{S})$ is polynomial iff $\mathcal{S}$ is a subset of some set in $\mathcal{R}_P$. Otherwise $\mathcal{R}$-SAT$(\mathcal{S})$ is NP-complete.

Proof:
$\Leftarrow$) For each $R \in \mathcal{R}_P$, $\mathcal{R}$-SAT(R) is polynomial by definition, and so are subsets of R.
$\Rightarrow$) Consider a set $\mathcal{S} \subseteq \mathcal{R}$ with $\mathcal{B} \subseteq \mathcal{S}$, $\mathcal{S}$ not being a subset of any set in $\mathcal{R}_P$. For each set C in $\mathcal{R}_P$, choose an element x such that $x \in \mathcal{S}$ and $x \notin C$. This can always be done since $\mathcal{S} \not\subseteq C$. Let X be the set of these elements. By the construction of X, $|X| \leq |\mathcal{R}_P|$. But then, by the condition of the theorem, either X is a subset of some set in $\mathcal{R}_P$, or $D \subseteq \mathcal{C}_{\mathcal{R}}(X \cup \mathcal{B})$ for some $D \in \mathcal{R}_{NP}$. But the former case cannot hold by the construction of X; thus $\mathcal{R}$-SAT$(\mathcal{C}_{\mathcal{R}}(X \cup \mathcal{B}))$ is NP-complete. It follows that $\mathcal{R}$-SAT$(X \cup \mathcal{B})$ is NP-complete, and since $X \cup \mathcal{B} \subseteq \mathcal{S}$, that $\mathcal{R}$-SAT$(\mathcal{S})$ is NP-complete. The result follows. $\Box$

We now proceed gradually with the classification by excluding certain combinations of basic relations. Note that the three conditions making Theorem 3.5 applicable always hold for Allen's algebra. Also note that any algebra has to contain an odd number of basic relations, since algebras are closed under the converse operation, and $(\equiv)$ is always included.

The following result is similar to one of Drakengren and Jonsson [1997].

Proposition 3.6 Let $A \subseteq \mathcal{A}$. If $(\mathsf{m}) \in A$, then either $A \subseteq \mathcal{H}$ or $\mathcal{A}$-SAT(A) is NP-complete.
Proof: It can easily be verified that $\mathbf{B} \subseteq \mathcal{C}_{\mathcal{A}}(\{(\mathsf{m})\})$ (use the `aclose` utility by Nebel and Bürckert [1993]), and the result follows by Proposition 3.4. $\Box$

Thus, $A \in \mathcal{T} \Rightarrow |bas(A)| \leq 11$.

Now for the first application of Theorem 3.5.

Proposition 3.7 Let $A \subseteq \mathcal{A}$. If $(\prec) \in bas(A)$, then either $A \subseteq \mathcal{H}$, $A \subseteq \mathcal{S}_{\prec}$, $A \subseteq \mathcal{E}_{\prec}$, or $\mathcal{A}$-SAT(A) is NP-complete.

Proof: First choose $\mathcal{R} = \mathcal{A}$, $\mathcal{R}_P = \{\mathcal{H}, \mathcal{S}_{\prec}, \mathcal{E}_{\prec}\}$, $\mathcal{R}_{NP} = \mathcal{A}_{NP}$ and $\mathcal{B} = \{(\prec)\}$. Then enumerate each set $T \subseteq \mathcal{A}$ with $|T| \leq |\mathcal{R}_P| = 3$ and test if for each T, either $T \subseteq \mathcal{H}$, $T \subseteq \mathcal{S}_{\prec}$, $T \subseteq \mathcal{E}_{\prec}$, or $D \subseteq \mathcal{C}_{\mathcal{A}}(T \cup \mathcal{B})$ for some $D \in \mathcal{A}_{NP}$. There are

$$\sum_{i=0}^{3} \binom{8192}{i} \approx 9.2 \cdot 10^{10}$$

such subsets. The test succeeds for all T, and the result follows. □

The subsets were enumerated on several Sun SPARC 10 stations in parallel, taking approximately 40 CPU weeks.

By this result, $A \in \mathcal{T} \Rightarrow |bas(A)| \leq 9$. The basic relations remaining to check are those in

$$Z = \{\mathsf{d}, \mathsf{d}^{\smile}, \mathsf{o}, \mathsf{o}^{\smile}, \mathsf{s}, \mathsf{s}^{\smile}, \mathsf{f}, \mathsf{f}^{\smile}\}.$$

If we can show that for any $r_1, r_2 \in Z$ with $r_1 \neq r_2$ and $r_1^{\smile} \neq r_2$, if for some $A \in \mathcal{T}$, $\{r_1, r_2\} \subseteq bas(A)$, then $A \subseteq \mathcal{H}$, $A \subseteq \mathcal{S}_r$ or $A \subseteq \mathcal{E}_r$ for some r, then we could conclude that $A \in \mathcal{T} \Rightarrow |bas(A)| \leq 3$, which is the goal of the paper. The following results will prove this.

Proposition 3.8 Let $A \subseteq \mathcal{A}$. For $W = \{(\mathsf{d}), (\mathsf{o})\}$ or $W = \{(\mathsf{s}), (\mathsf{f})\}$, if $W \subseteq bas(A)$, then either $A \subseteq \mathcal{H}$, or $\mathcal{A}$-SAT(A) is NP-complete.

Proof: First choose $\mathcal{R} = \mathcal{A}$, $\mathcal{R}_P = \{\mathcal{H}\}$, $\mathcal{R}_{NP} = \mathcal{A}_{NP}$ and $\mathcal{B} = W$. Then enumerate each set $T \subseteq \mathcal{A}$ with $|T| \leq |\mathcal{R}_P| = 1$ and test if for each T, either $T \subseteq \mathcal{H}$ or $D \subseteq \mathcal{C}_{\mathcal{A}}(T \cup \mathcal{B})$ for some $D \in \mathcal{A}_{NP}$. There are 8193 such subsets, regardless of W. The test succeeds for all T, and the result follows from Theorem 3.5. □

Proposition 3.9 Let $A \subseteq \mathcal{A}$. If $\{(\mathsf{s}), (r)\} \subseteq bas(A)$ for $r \in \{\mathsf{d}, \mathsf{o}\}$, then either $A \subseteq \mathcal{H}$, $A \subseteq \mathcal{S}_r$, or $\mathcal{A}$-SAT(A) is NP-complete.

Proof: First choose $\mathcal{R} = \mathcal{A}$, $\mathcal{R}_P = \{\mathcal{H}, \mathcal{S}_r\}$, $\mathcal{R}_{NP} = \mathcal{A}_{NP}$ and $\mathcal{B} = \{(\mathsf{s}), (r)\}$. Then enumerate each set $T \subseteq \mathcal{A}$ with $|T| \leq |\mathcal{R}_P| = 2$ and test if for each T, either $T \subseteq \mathcal{H}$, $T \subseteq \mathcal{S}_r$, or $D \subseteq \mathcal{C}_{\mathcal{A}}(T \cup \mathcal{B})$ for some $D \in \mathcal{A}_{NP}$. There are $\approx 3.4 \cdot 10^7$ such subsets. The test succeeds for all T, and the result follows from Theorem 3.5. □

The cases with $\{(\mathsf{f}),(\mathsf{d})\}$ and $\{(\mathsf{f}),(\mathsf{o})\}$ follow by symmetry from Proposition 3.9, using $\mathcal{E}_r$ instead of $\mathcal{S}_r$. We can thus conclude that $A \in \mathcal{T} \Rightarrow |bas(A)| \leq 3$, and that algebras in $\mathcal{T}$ can only contain basic relations in $\{\equiv, \mathsf{d}, \mathsf{o}, \mathsf{s}, \mathsf{f}\}$ which is the main result of the paper.

We conclude by a classification of all algebras containing the relation $(\prec\ \succ)$, needed for expressing the notion of *sequentiality*. This notion is important in many AI contexts, such as planning and reasoning about action [Sandewall, 1994], where actions are often assumed to come in sequence.

Proposition 3.10 Let $A \subseteq \mathcal{A}$. If $(\prec\ \succ) \in A$, then either $A \subseteq \mathcal{S}_\prec$, $A \subseteq \mathcal{E}_\prec$, or $\mathcal{A}$-SAT(A) is NP-complete.

Proof: First choose $\mathcal{R} = \mathcal{A}$, $\mathcal{R}_P = \{\mathcal{S}_\prec, \mathcal{E}_\prec\}$, $\mathcal{R}_{NP} = \mathcal{A}_{NP}$ and $\mathcal{B} = \{(\prec\ \succ)\}$. Then enumerate each set $T \subseteq \mathcal{A}$ with $|T| \leq |\mathcal{R}_P| = 2$ and test if for each T, either $T \subseteq \mathcal{S}_\prec$, $T \subseteq \mathcal{E}_\prec$, or $D \subseteq \mathcal{C}_\mathcal{A}(T \cup \mathcal{B})$ for some $D \in \mathcal{A}_{NP}$. There are $\approx 3.4 \cdot 10^7$ such subsets. The test succeeds for all T, and the result follows from Theorem 3.5. $\square$

Since both of these algebras also contain the relations $(\equiv)$, $(\prec)$, $(\succ)$, $(\equiv\ \prec)$, $(\equiv\ \succ)$, these are the only tractable algebras capable of expressing sequentiality.

In fact, when enumerating subsets in Proposition 3.7, Proposition 3.9 and Proposition 3.10, it is possible to optimise by stopping at subsets known to be NP-complete (those in $\mathcal{A}_{NP}$); sometimes with a factor of thirty.

4 Discussion

It is appropriate to indicate the applicability of this method to further classify tractability in $\mathcal{A}$. Therefore, consider the task of classifying all tractable algebras containing the basic relation (s). There are nine known maximal tractable algebras containing this relation. Thus, we have to enumerate all subsets of an 8192-element set having nine or fewer elements. This amounts to $4.6 \cdot 10^{29}$ subsets, making this task more difficult by a factor of 10^{19}, which is clearly impossible using today's computers.

For the full classification, we certainly need methods that combine theoretical studies of the structure of $\mathcal{A}$ with brute-force computer methods, similar to how the four-colour theorem was proved [Appel and Haken, 1976].

5 Conclusion

We have partially classified tractability of reasoning in Allen's interval algebra, with the result that any yet unknown tractable subclass can contain at most the basic relations $(\equiv)$, (b), $(b^{\smile})$, where $b \in \{\mathsf{d}, \mathsf{o}, \mathsf{s}, \mathsf{f}\}$. This means that for specifying complete knowledge about temporal relations, there is no hope of finding more expressive and yet tractable subclasses than those known today. Furthermore, we completely characterised the set of tractable subclasses which can express the notion of *sequentiality*, which is useful in many AI contexts.

Appendix

Here the intractability proofs needed for the proof of Proposition 3.2 are collected.

A.1 Model Transformations

Definition A.1 (Subsets Δ_0 and Δ_4) Let

$$R = (\mathsf{d}\ \mathsf{d}^{\smile}\ \mathsf{o}\ \mathsf{o}^{\smile})$$

as in Definition 3.1, and define

$$\Delta_0 = \{(\prec\ \succ), R \cup (\equiv\ \ \mathsf{m}\ \mathsf{m}^{\smile}\ \mathsf{s}\ \mathsf{s}^{\smile}\ \mathsf{f}\ \mathsf{f}^{\smile})\},$$

and

$$\Delta_4 = \{(\prec\ \succ), R \cup (\equiv\ \prec\ \ \mathsf{m}\ \mathsf{m}^{\smile}\ \mathsf{s}\ \mathsf{s}^{\smile}\ \mathsf{f}\ \mathsf{f}^{\smile})\}.$$

□

Proposition A.2 $\mathcal{A}$-SAT(Δ_0) and $\mathcal{A}$-SAT(Δ_4) are NP-complete.
Proof: See Golumbic and Shamir [1993]. □

The NP-completeness results of Proposition A.2 can be extended considerably by techniques introduced next.

Our main vehicle for showing intractability of different subclasses is that of *model transformations*. It is a method for transforming a solution of one problem to a solution of a related problem. The concept of model transformation and related results were introduced in the context of temporal reasoning in Jonsson *et al.* [1996].

Definition A.3 (Model transformation) A *model transformation* is a mapping on $\mathcal{A}$-interpretations. $\Box$

Next, a way to describe such transformations.

Definition A.4 (Model transformation description) Let T be a model transformation. A function $f_T : \mathbf{B} \to 2^{\mathbf{B}}$ is a *description* of T iff for arbitrary $\mathcal{A}$-interpretations $\Im$, the following holds: if $b \in \mathbf{B}$ and $I(b)J$ under $\Im$ then $I(f_T(b))J$ under $T(\Im)$. A description f_T can be extended to handle disjunctions in the obvious way: $f_T(R) = \bigcup_{r \in R} f_T(r)$. $\Box$

We can now provide a result on how model transformations can be used.

Lemma A.5 Let $\mathcal{R} = \{r_1, \ldots, r_n\} \subseteq \mathcal{A}$ and $\mathcal{R}' = \{r'_1, \ldots, r'_n\} \subseteq \mathcal{A}$ be such that $r'_k \subseteq r_k$ for all $1 \leq k \leq n$, and $\mathcal{A}$-SAT($\mathcal{R}$) is NP-complete. If there exists a model transformation T with a description f_T such that $f_T(r_k) \subseteq r'_k$ for every $1 \leq k \leq n$ then $\mathcal{A}$-SAT($\mathcal{R}'$) is NP-complete.
Proof: Let Θ be an instance of $\mathcal{A}$-SAT($\mathcal{R}$), and construct an instance Θ' of $\mathcal{A}$-SAT($\mathcal{R}'$) as follows by setting

$$\Theta' = \{Ir'_kJ \mid Ir_kJ \in \Theta\}.$$

We now prove that Θ is satisfiable iff Θ' is.
$\Leftarrow$) Suppose Θ' is satisfiable by a model $\Im'$. Then since every $Ir_kJ \in \Theta$ corresponds to an $Ir'_kJ \in \Theta'$ with $r'_k \subseteq r_k$, $\Im'$ is also a model of Θ.
$\Rightarrow$) Suppose Θ is satisfiable by a model $\Im$, and set $\Im' = T(\Im)$. We would like to verify that $\Im'$ is a model of Θ'. Take any $Ir_kJ \in \Theta$. Then $If_T(r_k)J$ holds in $\Im'$, by assumption. Since the r'_k corresponding to r_k satisfies $f_T(r_k) \subseteq r'_k$, also Ir'_k holds in $\Im'$, and the result follows. $\Box$

Before we define a model transformation that we will use later on, we need an auxiliary definition (also from Jonsson *et al.* [1996]).

Definition A.6 (Minimal distance) Let $S \subseteq \mathbb{R}$ be finite. The *minimal distance in* S, MD(S), is defined as

$$\min\{x - y \mid x, y \in S \wedge x > y\}.$$

$\Box$

Observe that $|S| \geq 2$ in order to make $\mathrm{MD}(S)$ defined. This is no problem, since we are working with intervals. For all such S, $\mathrm{MD}(S) > 0$. The definition of minimal distance can be extended to $\mathcal{A}$-interpretations in the following way: Let $\Im$ be an $\mathcal{A}$-interpretation that assigns values to a set of interval variables $\mathcal{I}$, and set

$$\mathrm{MD}(\Im) = \mathrm{MD}(\{\Im(I^-), \Im(I^+) \mid I \in \mathcal{I}\}).$$

A concrete model transformation follows.

Definition A.7 (Transformation T, description f) Define the model transformation T on $\mathcal{A}$-interpretations assigning values to interval variables $I_1, \ldots, I_n$ as follows. $T(\Im) = \Im'$, where $\Im'$ is obtained from $\Im$ by first setting $\epsilon = \mathrm{MD}(\Im)/(n+1)$ and defining

$$\Im'(I_i) = \langle \Im(I_i^-) - i\epsilon, \Im(I_i^+) + i\epsilon \rangle.$$

Then define $f(b)$ for $r \in \mathbf{B}$ as

$$f(\equiv) = \{\mathsf{d}, \mathsf{d}^{\smile}, \mathsf{o}, \mathsf{o}^{\smile}\},$$

$$f(\prec) = \{\prec\},$$

$$f(\succ) = \{\succ\},$$

$$f(\mathsf{d}) = \{\mathsf{d}\},$$

$$f(\mathsf{d}^{\smile}) = \{\mathsf{d}^{\smile}\},$$

$$f(\mathsf{o}) = \{\mathsf{o}\},$$

$$f(\mathsf{m}) = \{\mathsf{o}\},$$

$$f(\mathsf{m}^{\smile}) = \{\mathsf{o}^{\smile}\},$$

$$f(\mathsf{s}) = \{\mathsf{d}, \mathsf{o}\},$$

$$f(\mathsf{s}^{\smile}) = \{\mathsf{d}^{\smile}, \mathsf{o}^{\smile}\},$$

$$f(\mathsf{f}) = \{\mathsf{d}, \mathsf{o}^{\smile}\},$$

$$f(\mathsf{f}^{\smile}) = \{\mathsf{d}^{\smile}, \mathsf{o}\}.$$

□

Thus T decreases starting points and increases ending points of intervals, and does this differently for every interval. Now $f(b)$ represents what can happen to the basic relation b when the transformation T is applied.

Proposition A.8 f is a description of T.
Proof: Obvious from the definitions. □

We can now extend the results of Proposition A.2.

Theorem A.9 For any $A \in \mathcal{A}_0 \cup \mathcal{A}_4$ of Definition 3.1, $\mathcal{A}$-SAT(A) is NP-complete.
Proof: Let

$$r_1 = (\prec \ \succ)$$

and

$$r_2 = R \cup (\equiv \ \mathsf{m}\ \mathsf{m}^{\smile}\ \mathsf{s}\ \mathsf{s}^{\smile}\ \mathsf{f}\ \mathsf{f}^{\smile}),$$

where

$$R = (\mathsf{d}\ \mathsf{d}^{\smile}\ \mathsf{o}\ \mathsf{o}^{\smile}).$$

Thus

$$\mathcal{R} = \{r_1, r_2\} = \Delta_0$$

and is NP-complete. Take $A \in \mathcal{A}_0$. Now $A = \{r_1', r_2'\}$ for $r_1' = (\prec \ \succ)$ and

$$R \subseteq r_2' \subseteq R \cup (\equiv \ \mathsf{m}\ \mathsf{m}^{\smile}\ \mathsf{s}\ \mathsf{s}^{\smile}\ \mathsf{f}\ \mathsf{f}^{\smile}),$$

and it is obvious that the conditions of Lemma A.5 are satisfied. NP-completeness follows.

Similarly, let

$$r_1 = (\prec \ \succ)$$

and

$$r_2 = R \cup (\equiv \ \prec \ \mathsf{m}\ \mathsf{m}^{\smile}\ \mathsf{s}\ \mathsf{s}^{\smile}\ \mathsf{f}\ \mathsf{f}^{\smile}),$$

where

$$R = (\mathsf{d}\ \mathsf{d}^{\smile}\ \mathsf{o}\ \mathsf{o}^{\smile}).$$

Thus $\mathcal{R} = \{r_1, r_2\} = \Delta_4$ and is NP-complete. Take $A \in \mathcal{A}_4$. Now $A = \{r_1', r_2'\}$ for $r_1' = (\prec \ \succ)$ and

$$R \subseteq r_2' \subseteq R \cup (\equiv \ \prec \ \mathsf{m}\ \mathsf{m}^{\smile}\ \mathsf{s}\ \mathsf{s}^{\smile}\ \mathsf{f}\ \mathsf{f}^{\smile}),$$

and it is obvious that the conditions of Lemma A.5 are satisfied. NP-completeness follows. □

A.2 5-Composition

In order to find the last necessary NP-completeness results, we introduce a new operation to the Allen algebra.

Definition A.10 (5-composition) Let $r_1, \ldots, r_5 \in \mathcal{A}$. Define the *5-composition* of $r_1, \ldots, r_5$, denoted $\mathit{5comp}(r_1, \ldots, r_5)$, by

$$I \ \mathit{5comp}(r_1, \ldots, r_5) \ J \Leftrightarrow \exists K, L. I r_1 K,\ K r_2 J,\ I r_3 L,\ L r_4 J,\ K r_5 L.$$

□

The 5-composition of relations $r_1, \ldots, r_5$ can easily be computed by using Nebel's software for computing satisfiability of networks of Allen relations[1], by constructing a network of four interval variables with relations according to the definition, and computing the entailed relation between two of the variables by choosing the basic relations which are consistent there.

NP-completeness results can be obtained as follows.

Proposition A.11 Let $A \subseteq \mathcal{A}$, and suppose $\mathcal{A}$-SAT$(A \cup \{\mathit{5comp}(r_1, \ldots, r_5)\})$ can be shown to be NP-complete, for $r_i \in A$. Then $\mathcal{A}$-SAT(A) is NP-complete.
Proof: Any network expressed using the extended set of relations can be converted to an equivalent one using only relations from A, by the definition of 5-composition. The transformation is obviously polynomial. □

Theorem A.12 $\mathcal{A}$-SAT for the subclasses $\mathcal{N}_3$, $\mathcal{N}_4$ and $\mathcal{N}_5$ are NP-complete.

Proof: Recall the definitions:

$$\mathcal{N}_3 = \{(\prec \ \succ), (\mathsf{o} \ \mathsf{o}^\smile)\},$$

$$\mathcal{N}_4 = \{(\prec \ \succ), (\mathsf{o} \ \mathsf{o}^\smile \ \mathsf{m} \ \mathsf{m}^\smile)\}$$

and

$$\mathcal{N}_5 = \{(\mathsf{m} \ \mathsf{m}^\smile), (\prec \ \succ \ \mathsf{s} \ \mathsf{s}^\smile \ \mathsf{f} \ \mathsf{f}^\smile)\}.$$

Define

$$c(r_1, r_2) = \mathit{5comp}(r_1, r_1, r_1, r_1, r_2).$$

[1]This software was developed for obtaining the results of Nebel's paper [1996], and can be obtained from Bernhard Nebel.

First, we verify that

$$r_3 = c((\mathsf{o}\ \mathsf{o}^\smile), (\prec\ \succ)) = (\equiv\ \ \mathsf{d}\ \mathsf{d}^\smile\ \mathsf{o}\ \mathsf{o}^\smile\ \mathsf{s}\ \mathsf{s}^\smile\ \mathsf{f}\ \mathsf{f}^\smile),$$

and we see that $A \subseteq \mathcal{N}_3 \cup \{r_3\}$ for some $A \in \mathcal{A}_0$, and NP-completeness follows by Proposition 3.2. Next,

$$r_4 = c((\mathsf{o}\ \mathsf{o}^\smile\ \ \mathsf{m}\ \mathsf{m}^\smile), (\prec\ \succ)) = (\equiv\ \ \mathsf{d}\ \mathsf{d}^\smile\ \mathsf{o}\ \mathsf{o}^\smile\ \mathsf{s}\ \mathsf{s}^\smile\ \mathsf{f}\ \mathsf{f}^\smile),$$

and we see that $A \subseteq \mathcal{N}_4 \cup \{r_4\}$ for some $A \in \mathcal{A}_0$, and NP-completeness follows by Proposition 3.2. Last,

$$r_5 = c((\mathsf{m}\ \mathsf{m}^\smile), (\prec\ \succ\ \ \mathsf{s}\ \mathsf{s}^\smile\ \mathsf{f}\ \mathsf{f}^\smile)) = (\equiv\ \ \mathsf{s}\ \mathsf{s}^\smile\ \mathsf{f}\ \mathsf{f}^\smile),$$

and it can be verified using Nebel and Bürckert's software [1993] that $\Delta_0 \subseteq \mathcal{C}_{\mathcal{A}}(\mathcal{N}_5 \cup \{r_5\})$, implying NP-completeness by Proposition 3.2 and Proposition 2.4. $\square$

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Paper V

Reasoning about Action in Polynomial Time

Thomas Drakengren and Marcus Bjäreland

ABSTRACT

Although many formalisms for reasoning about action exist, surprisingly few approaches have taken computational complexity into consideration. The contributions of this paper are the following: a temporal logic with a restriction for which deciding satisfiability is tractable, a tractable extension for reasoning about action, and NP-completeness results for the unrestricted problems. Many interesting reasoning problems can be modelled, involving nondeterminism, concurrency and memory of actions. The reasoning process is proved to be sound and complete.

1 Introduction

Although many formalisms for reasoning about action exist, surprisingly few approaches have taken computational complexity into consideration. One explanation for this might be that many interesting AI problems are (at least) NP-hard, and that tractable subproblems that are easily extracted, tend to lack expressiveness. This has led a large part of the AI community to rely on heuristics and incomplete systems to solve the problems (see, for example, [Ginsberg, 1996] for a discussion). This holds, in particular, for the area of reasoning about action, where the very expressive logical formalisms provide difficult obstacles when it comes to efficient implementation.

We feel, however, that the tractability boundary for sound and complete reasoning about action has not yet been satisfactorily investigated. We prove this by introducing a nontrivial subset of a logic with semantics closely related to the *trajectory semantics* of Sandewall [1994], for which satisfiability is tractable. Our logic can handle examples involving not only nondeterminism, but continuous time, concurrency and memory of actions as well, thus providing a conceptual extension of Sandewall's framework. The reader should note that our main concern is *computation*, as opposed to *modelling*.

This paper is organised as follows: Section 2 is an informal overview of the technical results, where two examples are also presented. In Section 3 we present the syntax and semantics of the basic temporal logic and the extension for reasoning about action, and in Section 4, we present the following: three intractability results for these formalisms, and the main results: the tractable subclasses of the temporal logic and its extension.

2 Overview

In Section 3 we develop a temporal logic, Λ, which is syntactically related to the propositional temporal logic TPTL [Alur and Henzinger, 1989]. The temporal domain is the set of real numbers and temporal expressions are based on relations $=$, $\leq$, $<$, $\geq$ and $>$ between linear polynomials with rational coefficients over a set of temporal variables. The semantics of this temporal logic is standard. The formalism for reasoning about action is *narrative* based, which means that *scenario descriptions* are used to model the real world. Scenario descriptions consist of formulae in the temporal logic (*observations*) and *action expressions* which are constructs that state that certain changes in values of the *features* (propositions, fluents) may occur.

We write action expressions as $\pi \Rrightarrow [\alpha]\epsilon\, \texttt{Infl}$, where π is the precondition for the action, ϵ the effects, α a temporal expression denoting *when* the effects are taking place, and $\texttt{Infl}$ is the set of all features that are influenced by the action. The influenced features are not subject to the assumption of inertia, i.e. we allow them, and only them, to change during the execution of the action.

It turns out that deciding satisfiability is NP-complete, both for the temporal logic and the scenario descriptions. Interestingly, the problem is NP-complete for scenario descriptions that only include Horn clause observations, unconditional and unary action expressions (this terminology is explained later), and no stated relations between temporal expressions.

To extract a tractable subset from our formalism we rely on a recent result in *temporal constraint reasoning* by Jonsson and Bäckström [1996] (also discovered independently by Koubarakis [1996]). They have identified a large tractable class of temporal constraint reasoning, using *Horn Disjunctive Linear Relations* (Horn DLRs) which are relations between linear polynomials with rational coefficients. We make use of their result by restricting formulae in our scenario descriptions to be Horn and then by encoding scenario descriptions into Horn DLRs. For the temporal logic this is fairly straightforward. For the scenario descriptions, it turns out that we have to put some constraints on the temporal relations and actions in the scenario descriptions.

We will use the following two examples: Jump into a Lake with a Hat [Giunchiglia and Lifschitz, 1995] and Soup Bowl Lifting [Gelfond *et al.*, 1991]. Below we informally describe the examples.

Example 1 (Jump into a Lake with a Hat, JLH) If you jump into the lake you will get wet. If you have been in the water at some time point it is unclear if you still have your hat on. This is an example of *nondeterminism* and of *memory of actions.* □

Example 2 (Soup Bowl Lifting, SBL) If we lift either side of a soup bowl at some time points, the content will be spilled, unless we lift both sides at the same time point. This is an example of *concurrency.* □

The first example can be handled by the tractable subset of our formalism, whereas the second one fails to satisfy the restrictions. Note that actions may partly overlap.

3 Scenario Descriptions

We introduce a semantics that is a simpler variant of Sandewall's Features and Fluents Framework [Sandewall, 1994], in that the effects of an action can only occur at one and the same time point for a given action, and we use only propositional values of features (similar to the work of Doherty [1994]) However, in some respects this formalism is more flexible than Sandewall's: we use a *continuous* time domain, we allow *concurrently* executing actions, and effects of actions can depend on other states in the history than the state at the *starting time point* of the action (this implies *memory* of actions, in Sandewall's [1994] terminology). One example of a formalism having memory is that of Gustafsson and Doherty [1996].

Initially, a basic temporal logic is defined. The computational properties of this logic will be exploited by the scenario description logic, i.e. ultimately (in Section 4) the scenario descriptions will be transformed into formulae of the basic temporal logic.

3.1 Syntax

We begin by defining the basic temporal logic.

We assume that we have a set $\mathcal{T}$ of time point variables intended to take real values, and a set $\mathcal{F}$ of *features* intended to take propositional values.

Definition 3 A *signature* is a tuple $\sigma = \langle \mathcal{T}, \mathcal{F} \rangle$, where $\mathcal{T}$ is a finite set of time point variables and $\mathcal{F}$ is a finite set of propositional features. A *time point expression* is a linear polynomial over $\mathcal{T}$ with rational coefficients. We denote the set of time point expressions over $\mathcal{T}$ by $\mathcal{T}^*$. $\square$

Definition 4 Let $\sigma = \langle \mathcal{T}, \mathcal{F} \rangle$ be a signature, let $\alpha, \beta \in \mathcal{T}^*$, $f \in \mathcal{F}$, $R \in \{=, \leq, <, \geq, >\}$, $\oplus \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$, and define the *scenario description language* Λ over σ by

$$\Lambda ::= \mathtt{T} \mid \mathtt{F} \mid f \mid \alpha R \beta \mid \neg\Lambda \mid \Lambda_1 \oplus \Lambda_2 \mid [\alpha]\Lambda.$$

A formula of the form $\alpha R \beta$ is a *linear relation*, and one that does not contain any connectives (any of the constructs $\wedge, \vee, \rightarrow, \leftrightarrow, \neg$ and $[\cdot]$) is *atomic*. If γ is atomic and $\alpha \in \mathcal{T}^*$, then the formulae γ, $[\alpha]\gamma$, $\neg\gamma$, $[\alpha]\neg\gamma$ and $\neg[\alpha]\gamma$ are *literals*[1]. (A formula $[\alpha]\gamma$ expresses that at time α, γ is true.) A literal l is *negative* iff it contains $\neg$ and its corresponding atomic formula γ is not of

[1] So, the formula $\neg[\alpha]\neg\gamma$ is not a literal.

the form $\alpha R \beta$ for $R \in \{<, \leq, >, \geq\}$. A literal that is not negative is *positive.* Disjunctions of literals are *clauses.* A formula $\gamma \in \Lambda$ is in *conjunctive normal form*, CNF, iff it is a conjunction of clauses. A formula γ is *Horn* iff it is a clause with at most one positive literal. A set Γ of formulae is *Horn* iff every $\gamma \in \Gamma$ is Horn. Syntactical identity between formulae is written $\equiv$, and when ambiguity is to be avoided, we denote formulae $\gamma \in \Lambda$ by $\ulcorner\gamma\urcorner$.

Let γ be a formula. A feature $f \in \mathcal{F}$ occurs *free* in γ iff it does not occur within the scope of a $[\alpha]$ expression in γ. $\alpha \in \mathcal{T}^*$ *binds* f in γ if a formula $[\alpha]\phi$ occurs as a subformula of γ, and f is free in ϕ. If no feature occurs free in γ, γ is *closed.* If γ does not contain any occurrence of $[\alpha]$ for any $\alpha \in \mathcal{T}^*$, then γ is *propositional.* $\Box$

Using Λ, we can thus express propositions being true at time points, and express relations between time points. Next we define the extension of the basic temporal logic by introducing *action expressions*, i.e. constructs that enable modelling of change.

Definition 5 Let $\sigma = \langle \mathcal{T}, \mathcal{F} \rangle$ be a signature. An *action expression* over σ is a tuple $A = \langle \alpha, \pi, \mathtt{Infl}, \epsilon \rangle$, $\alpha \in \mathcal{T}^*$, π a closed formula in Λ, $\mathtt{Infl} \subseteq \mathcal{F}$, and ϵ a propositional formula, where all features occurring in ϵ are in $\mathtt{Infl}$. α is the *result time point* of A, π is the *precondition* of A, $\mathtt{Infl}$ is the set of *influenced features* of A, and ϵ is the *effects* of A. A is *unconditional* iff $\pi \equiv \mathtt{T}$, and *unary* iff $|\mathtt{Infl}| = 1$.

For convenience we write action expressions as

$$\pi \Rrightarrow [\alpha]\epsilon\, \mathtt{Infl};$$

for example we have

$$[3] loaded \Rrightarrow [4]\neg alive\{alive\}$$

for an action shoot. If $\pi \equiv \mathtt{T}$, we remove it and the $\Rrightarrow$ symbol, and if $\epsilon \equiv \mathtt{T}$, we remove it. An example is an unconditional loading action

$$[2] loaded\{loaded\},$$

and the action of spinning the chamber of a gun

$$[3]\{loaded\}.$$

An *observation* over σ is a closed formula in Λ. $\Box$

Next, we combine the concepts defined so far into one.

Definition 6 A *scenario description* is a tuple

$$\Upsilon = \langle \sigma, \textsc{scd}, \textsc{obs} \rangle,$$

where $\sigma = \langle \mathcal{T}, \mathcal{F} \rangle$ is a signature, SCD (the *schedule*) is a finite set of action expressions over σ, and OBS is a finite set of observations over σ. The *size* of a scenario description is defined as the sum of lengths of all formulae in SCD and OBS. □

Now, we formalise the examples from Section 2.

Example 7 (JLH) The intended conclusion of the following scenario is that the person is wet at time $\mathbf{c}_1$, and we do not know if the hat is on at time point $\mathbf{c}_2$, occurring after the person jumps.

OBS1	$[0]hat\_on \wedge dry \wedge on\_land$
SCD1	$[\mathbf{c}_1]\neg on\_land\{on\_land\}$
SCD2	$[\mathbf{c}_1]\neg on\_land \Rrightarrow [\mathbf{c}_1]\neg dry\{dry\}$
SCD3	$[\mathbf{c}_1]\neg on\_land \Rrightarrow [\mathbf{c}_2]\{hat\_on\}$
OBS4	$\mathbf{c}_1 \geq 0 \wedge \mathbf{c}_2 \geq \mathbf{c}_1$

□

Example 8 (SBL) We have two actions: one for lifting the left side of the soup bowl and one for lifting the right side. If the actions are not executed simultaneously, the table cloth will no longer be dry. The intended conclusion here is $\mathbf{c}_2 = \mathbf{c}_1$.

OBS1	$[0]dry$
SCD1	$[\mathbf{c}_1]leftup\{leftup\}$
SCD2	$[\mathbf{c}_1]\neg rightup \Rrightarrow [\mathbf{c}_1]\neg dry\{dry\}$
SCD3	$[\mathbf{c}_2]rightup\{rightup\}$
SCD4	$[\mathbf{c}_2]\neg leftup \Rrightarrow [\mathbf{c}_2]\neg dry\{dry\}$
OBS2	$[\mathbf{c}_2]dry$
OBS3	$\mathbf{c}_2 > 0 \wedge \mathbf{c}_1 > 0$

□

3.2 Semantics

For the presentation of the semantics we proceed similarly to the presentation of the syntax. We begin by defining the semantics of the basic temporal logic.

Definition 9 Let $\sigma = \langle \mathcal{T}, \mathcal{F} \rangle$ be a signature. A *state* over σ is a function from $\mathcal{F}$ to the set $\{\mathtt{T}, \mathtt{F}\}$ of truth values. A *history* over σ is a function h from $\mathbb{R}$ to the set of states. A *valuation* ϕ is a function from $\mathcal{T}$ to $\mathbb{R}$. It is extended in a natural way (as a homomorphism from $\mathcal{T}^*$ to $\mathbb{R}$), giving e.g.

$$\phi(3t + 4.3) = 3\phi(t) + 4.3.$$

A *development*, or *interpretation*, over σ is a tuple $\langle h, \phi \rangle$ where h is a history and ϕ is a valuation. □

Definition 10 Let $\gamma \in \Lambda$, and let $D = \langle h, \phi \rangle$ be a development. Define the *truth value* of γ in D for a time point $t \in \mathbb{R}$, denoted $D(\gamma, t)$, as follows (here we overload $\mathtt{T}$ and $\mathtt{F}$ to denote both formulae and truth values). Assume $f \in \mathcal{F}$, $R \in \{=, \leq, <, \geq, >\}$, $\alpha, \beta \in \mathcal{T}^*$, $\gamma, \delta \in \Lambda$, $\oplus \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$, and $\tau \in \{\mathtt{T}, \mathtt{F}\}$. Now define

$$\begin{array}{lcl}
D(\tau, t) & = & \tau \\
D(f, t) & = & h(t)(f) \\
D(\alpha R \beta, t) & = & \phi(\alpha) R \phi(\beta) \\
D(\neg\gamma, t) & = & \neg D(\gamma, t) \\
D(\gamma \oplus \delta, t) & = & D(\gamma, t) \oplus D(\delta, t) \\
D([\alpha]\gamma, t) & = & D(\gamma, \phi(\alpha)).
\end{array}$$

Two formulae γ_1 and γ_2 are *equivalent* iff $D(\gamma_1, t) = D(\gamma_2, t)$ for all D and t. A set $\Gamma \subseteq \Lambda$ of formulae is *satisfiable* iff there exists a development D and a time point $t \in \mathbb{R}$ such that $D(\gamma, t)$ is true for every $\gamma \in \Gamma$. A development D is a *model* of a set $\Gamma \subseteq \Lambda$ of closed formulae iff $D(\gamma, t)$ is true for every $t \in \mathbb{R}$ and $\gamma \in \Gamma$. □

Fact 11 For $\gamma \in \Lambda$ and $\alpha \in \mathcal{T}^*$, $\neg[\alpha]\gamma$ is equivalent to $[\alpha]\neg\gamma$. For a closed formula γ, $D(\gamma, t) = D(\gamma, t')$ for any $t, t' \in \mathbb{R}$ and development D. □

Thus, if γ is closed, we can write $D(\gamma)$ instead of $D(\gamma, t)$.

Now we define the semantics of the action expressions based on models for the basic temporal logic. Inertia (the frame problem) is handled by identifying all time points where a feature f possibly can change its value. Then during every interval where no such change time point exists, f has to have the same value throughout the interval.

Definition 12 Let $D = \langle h, \phi \rangle$ be a development. An action expression $A = \langle \alpha, \pi, \texttt{Infl}, \epsilon \rangle$ *has effects* in D iff $D(\pi) = \texttt{T}$. Let $f \in \mathcal{F}$, and define $Chg(D, \textsc{scd}, f, t)$ to be true for a time point $t \in \mathbb{R}$ iff $f \in \texttt{Infl}$ for some action expression $A = \langle \alpha, \pi, \texttt{Infl}, \epsilon \rangle \in \textsc{scd}$ that has effects in D, with $\phi(\alpha) = t$. Note that we can only have $Chg(D, \textsc{scd}, f, t)$ true for a finite number of time points for fixed $\textsc{scd}$ and f.

Let $\Upsilon = \langle \sigma, \textsc{scd}, \textsc{obs} \rangle$ be a scenario description. An *intended model* of Υ is a development $D = \langle h, \phi \rangle$ where

- D is a model of $\textsc{obs}$
- For each $A = \langle \alpha, \pi, \texttt{Infl}, \epsilon \rangle \in \textsc{scd}$ that has effects in D, $D(\epsilon, \phi(\alpha)) = \texttt{T}$
- For each $f \in \mathcal{F}$ and $s, t \in \mathbb{R}$ with $s < t$ such that for no $t' \in (s, t)$ (open interval), $Chg(D, \textsc{scd}, f, t')$ holds, we have $h(t')(f) = h(s)(f)$ for every $t' \in (s, t)$.

Denote by $Mod(\Upsilon)$ the set of all intended models for a scenario description Υ.

A formula $\gamma \in \Lambda$ is *entailed* by a scenario description Υ, denoted $\Upsilon \models \gamma$, iff γ is true in all intended models of Υ. Υ is *satisfiable* iff $Mod(\Upsilon) \neq \emptyset$. □

Fact 13 If $\Upsilon = \langle \sigma, \textsc{scd}, \textsc{obs} \rangle$ is a scenario description and $\gamma \in \Lambda$ a formula, then $\Upsilon \models \gamma$ iff $\langle \sigma, \textsc{scd}, \textsc{obs} \cup \{\neg\gamma\} \rangle$ is unsatisfiable. □

We comment how this is used in our two examples:

- It is intended that JLH is unsatisfiable adding the observation $[\mathbf{c}_1] dry$, and that neither adding $[\mathbf{c}_2] hat\_on$ nor $[\mathbf{c}_2] \neg hat\_on$ will make it unsatisfiable.
- For SBL, adding $\mathbf{c}_2 \neq \mathbf{c}_1$ as an observation will make the scenario unsatisfiable.

4 Complexity Results

4.1 Basic Results

It is no surprise that deciding satisfiability for the basic temporal logic is NP-hard. Proofs of NP-completeness, on the other hand, depend on the tractability results.

Proposition 14 Deciding satisfiability of a set $\Gamma \subseteq \Lambda$ is NP-hard.
Proof: Propositional logic is a subset of Λ. $\Box$

Corollary 15 Deciding whether a scenario description is satisfiable is NP-hard. $\Box$

That these problems are in NP, and thus are NP-complete, is proved in Theorem 23 and Theorem 24.

Interestingly, we can strengthen the result considerably.

Theorem 16 Deciding whether a scenario description is satisfiable is NP-hard, even if action expressions are unconditional and unary, only Horn observations are allowed, and no disjunctive relations between time points may be stated.
Proof: Reduction from 3SAT. Let $C_i = l_{i1} \vee l_{i2} \vee l_{i3}$ be the clauses of a 3CNF formula ϕ, and v the set of propositional symbols used in ϕ. We construct a scenario description Υ satisfying the required restrictions as follows. Set

$$\mathcal{F} = v \cup \{p' \mid p \in v\},$$

$$\mathcal{T} = \{s, t\} \cup \{t_p \mid p \in v\},$$

$$\text{SCD} = \{\langle t, \mathtt{T}, \{p\}, \neg p\rangle \mid p \in v\} \cup \{\langle t, \mathtt{T}, \{p'\}, p'\rangle \mid p \in v\},$$

$$\begin{aligned}\text{OBS} = \ & \{t > s\} \cup \{[s]p, [s]\neg p' \mid p \in v\} \cup \\ & \{\mathcal{C}(l_{i1}) \vee \mathcal{C}(l_{i2}) \vee \mathcal{C}(l_{i3})\},\end{aligned}$$

where $\mathcal{C}(p) = \neg[t_p]p'$ and $\mathcal{C}(\neg p) = \neg[t_p]p$.

We shall use intended models D as models for ϕ, interpreting $D(t_p < t)$ as the truth value of p, and so show that Υ is satisfiable iff ϕ is satisfiable. First note the facts that in any $D \in \mathit{Mod}(\Upsilon)$, $D(\neg[t_p]p') = D([t_p]p)$, and that $D([t_p]p) = D(t_p < t)$.
$\Rightarrow$) Suppose $D \in \mathit{Mod}(\Upsilon)$, and consider a clause $C_i = l_{i1} \vee l_{i2} \vee l_{i3}$ in ϕ. By the construction of OBS, one of $\mathcal{C}(l_{i1})$, $\mathcal{C}(l_{i2})$ or $\mathcal{C}(l_{i3})$ has to be true in D, say $l = l_{i1}$. If $l = p$, then $\mathcal{C}(l) = \neg[t_p]p'$; thus $D([t_p]p)$ is true, and so is $D(t_p < t)$. If $l = \neg p$, then $\mathcal{C}(l) = \neg[t_p]p$; thus $D([t_p]p)$ is false, and so is $D(t_p < t)$.
$\Leftarrow$) Suppose ϕ has a propositional model M, and consider a clause $C_i = l_{i1} \vee l_{i2} \vee l_{i3}$ in ϕ. Construct an intended model D of Υ by letting features have values as forced by the scenario, t having the value 0 and s the value -1, and for each $p \in v$, if p is true in M, then set $t_p = -1$, and otherwise $t_p = 1$. It is clear that the expression $\mathcal{C}(l_{i1}) \vee \mathcal{C}(l_{i2}) \vee \mathcal{C}(l_{i3})$ is true in D. The result follows, since it is clear that the reduction is polynomial. $\Box$

We now present the key to tractability, which is a linear-programming approach to temporal constraint reasoning, by Jonsson and Bäckström [1996].

Definition 17 Let α and β be linear polynomials with rational coefficients over some set X of variables. Then a *disjunctive linear relation*, DLR, is a disjunction of one or more expressions of the form $\alpha = \beta$, $\alpha \neq \beta$, $\alpha \leq \beta$, $\alpha < \beta$. A DLR is *Horn* iff it contains at most one disjunct with the relation $=$, $<$ or $\leq$.

An assigment m of variables in X to real numbers is a *model* of a set Γ of DLRs iff all formulae in Γ are true when taking the values of variables in the DLRs. A set of DLRs is *satisfiable* iff it has a model. $\Box$

The following result is the main result of Jonsson and Bäckström [1996].

Proposition 18 Deciding satisfiability of a set of Horn DLRs is polynomial. $\Box$

Now we restrict the scenario description language and the form of actions. Furthermore, a structural restriction on scenario descriptions, verifiable in polynomial time, is imposed. We shall define an encoding function that takes a Horn scenario description Υ and returns a set Γ of Horn DLRs such that Γ is satisfiable iff Υ is satisfiable.

4.2 Satisfiability of Horn Formulae is Tractable

First, we code Horn formulae as Horn DLRs.

Definition 19 Let l be a closed literal, and assume the existence of fresh, unique time point variables t_f^α for each $f \in \mathcal{F}$ and $\alpha \in \mathcal{T}^*$. Then $C(l)$ is defined as follows, assuming $R \in \{=, \leq, <, \geq, >\}$ and $f \in \mathcal{F}$.

$$
\begin{array}{lcl}
C(\mathtt{T}) & = & \ulcorner 0 = 0 \urcorner \\
C(\mathtt{F}) & = & \ulcorner 0 \neq 0 \urcorner \\
C(\alpha R \beta) & = & \ulcorner \alpha R \beta \urcorner \\
C(\neg \alpha = \beta) & = & \ulcorner \alpha \neq \beta \urcorner \\
C(\neg \alpha \leq \beta) & = & \ulcorner \alpha > \beta \urcorner \\
C(\neg \alpha < \beta) & = & \ulcorner \alpha \geq \beta \urcorner \\
C(\neg \alpha \geq \beta) & = & \ulcorner \alpha < \beta \urcorner \\
C(\neg \alpha > \beta) & = & \ulcorner \alpha \leq \beta \urcorner \\
C([\eta] \alpha R \beta) & = & C(\alpha R \beta) \\
C([\alpha] f) & = & \ulcorner t_f^\alpha = 0 \urcorner \\
C(\neg [\alpha] \gamma) & = & C([\alpha] \neg \gamma) \\
C([\alpha] \neg f) & = & \ulcorner t_f^\alpha \neq 0 \urcorner
\end{array}
$$

Let $\gamma \in \Lambda$ be a closed Horn formula, and let γ' be obtained from γ by simplifying away occurrences of T and F. Now γ' is on the form $\bigvee_i l_i$. Then define $C(\gamma)$ to be the DLR $\delta = \bigvee_i C(l_i)$. Note that δ is always a Horn DLR.

Let $\Gamma \subseteq \Lambda$ be a set of closed Horn formulae, and T the set of all time point expressions occurring in Γ. Then $C(\Gamma)$ is defined by

$$C(\Gamma) = \{C(\gamma) \mid \gamma \in \Gamma\} \cup \{C(\neg[\alpha]f \vee \beta \neq \alpha \vee [\beta]f) \mid f \in \mathcal{F}, \alpha, \beta \in T\}.$$

The second set is called the *correspondence equations*. Note that the argument of C in a correspondence equation is equivalent to

$$[\alpha]f \wedge \beta = \alpha \rightarrow [\beta]f.$$

□

The following result is crucial.

Theorem 20 Let $\Gamma \subseteq \Lambda$ be a set of closed Horn formulae. Then Γ is satisfiable iff $C(\Gamma)$ is satisfiable.
Proof: Let T be the set of time point expressions occurring in Γ.
$\Rightarrow$) Let $D = \langle h, \phi \rangle$ be a model of Γ. We shall construct a model m of $C(\Gamma)$. First set $m(t) = \phi(t)$ for all $t \in \mathcal{T}$. Now all temporal relations from Γ are satisfied in m, since they are directly transferred. For each $f \in \mathcal{F}$ and $\alpha \in T$, if $h(\phi(\alpha))(f)$ is true, then set $m(t_f^\alpha) = 0$, otherwise set $m(t_f^\alpha) = 1$. It is clear that the correspondence equations are satisfied by this definition, and so are the remaining elements of $C(\Gamma)$.
$\Leftarrow$) Let m be a model of $C(\Gamma)$, Construct an interpretation $D = \langle h, \phi \rangle$ as follows. First set $\phi(t) = m(t)$ for all $t \in \mathcal{T}$. It is enough to determine h for values $\phi(\alpha)$, $\alpha \in T$, since we have no restrictions on other values. Set $h(\phi(\alpha))(f)$ to be true iff $m(t_f^\alpha) = 0$. That h is well-defined follows directly from the correspondence equations, which hold in m. □

Corollary 21 Deciding satisfiability of sets of closed Horn formulae is polynomial.
Proof: It is clear that the transformation C is polynomial. The result follows from Proposition 18. □

4.3 Satisfiability is in NP

Now we have the results for the proofs of membership in NP for the satisfiability problems of Λ and of scenario descriptions. First, we need an auxiliary definition.

Definition 22 Let $\Gamma \subseteq \Lambda$. Then we define $\neg\Gamma = \{\neg\gamma \mid \gamma \in \Gamma\}$, $\mathit{feat}(\Gamma)$ to be the set of all features occurring in Γ, $\mathit{linrel}(\Gamma)$ to be the set of all linear relations (which are atomic formulae) occurring in Γ, and $\mathit{time}(\Gamma)$ to be the set of all time point expressions occurring in Γ. $\Box$

Theorem 23 Deciding satisfiability of a set $\Gamma \subseteq \Lambda$ is NP-complete.
Proof: By Proposition 14, it remains to prove that the problem is in NP. Let $\Gamma \subseteq \Lambda$ be a set of formulae. A few auxiliary definitions will be needed for the proof.

Let $F = \mathit{feat}(\Gamma)$, $L = \mathit{linrel}(\Gamma)$, $T = \mathit{time}(\Gamma)$, let $\mathtt{t}$ be a fresh time point variable, set

$$A^+ = \{[\alpha]f \mid \alpha \in T \cup \{\mathtt{t}\} \wedge f \in F\} \cup L,$$

and

$$A' = A^+ \cup \neg A^+.$$

We say that a set $W \subseteq A'$ is a *syntactic* Λ*-interpretation* iff for any $\phi \in A^+$, exactly one of ϕ and $\neg\phi$ is a member of W.

Next, we define a way to evaluate a formula $\gamma \in \Lambda$ in a syntactic Λ-interpretation, following Definition 10. Let $\gamma \in \Lambda$. We define the *truth value* of γ in W for $\alpha' \in \mathcal{T}^*$, denoted $W(\gamma, \alpha')$, as follows. Assume $f \in \mathcal{F}$, $R \in \{=, \leq, <, \geq, >\}$, $\alpha, \beta \in \mathcal{T}^*$, $\gamma, \delta \in \Lambda$, and $\oplus \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$.

- $W(\mathtt{T}, \alpha') = \mathtt{T}$
- $W(\mathtt{F}, \alpha') = \mathtt{F}$
- $W(f, \alpha') = (\ulcorner[\alpha']f\urcorner \in W)$
- $W(\alpha R \beta, \alpha') = (\ulcorner\alpha R \beta\urcorner \in W)$
- $W(\neg\gamma, \alpha') = \neg W(\gamma, \alpha')$
- $W(\gamma \oplus \delta, \alpha') = W(\gamma, \alpha') \oplus W(\delta, \alpha')$
- $W([\alpha]\gamma, \alpha') = W(\gamma, \alpha)$

It is clear that $W(\gamma, \alpha')$ is always defined for $\gamma \in \Lambda$. Thus we can say that a syntactic Λ-interpretation W is a *syntactic* Λ*-model* of a set Γ of formulae iff W is satisfiable, and for all $\gamma \in \Gamma$, $W(\gamma, \mathtt{t})$ is true.

If we can prove that whenever Γ is satisfiable, then there exists a syntactic Λ-model W of Γ, and vice versa, then NP-membership will follow, since

- the size of W is polynomial in the size of Γ
- it can be checked in polynomial time whether W is a syntactic Λ-interpretation or not
- $W(\gamma, \alpha')$ can easily be computed in polynomial time
- by Corollary 21, checking satisfiability of a set of closed Horn formulae is polynomial, and thus checking satisfiability of W is polynomial.

$\Rightarrow$) Suppose that Γ is satisfiable, i.e. that for some $t \in \mathbb{R}$, $I(\gamma, t) = \mathtt{T}$ for all $\gamma \in \Gamma$. Construct a syntactic Λ-interpretation W from $I = \langle h, \phi \rangle$ by first setting $I' = \langle h, \phi' \rangle$, where $\phi'(s) = \phi(s)$ for all $s \in \mathcal{T} - \{\mathtt{t}\}$, and $\phi'(\mathtt{t}) = t$, and then defining

$$W = \{\delta \in A' \mid I'(\delta) = \mathtt{T}\}.$$

It remains to check that W is satisfiable, and that $W(\gamma, \mathtt{t}) = \mathtt{T}$ for all $\gamma \in \Gamma$.

W is trivially satisfiable, since by construction I' is a model of W. We prove by induction on γ that $W(\gamma, \alpha') = I(\gamma, \phi'(\alpha'))$ for all $\gamma \in \Lambda$ and $\alpha' \in T \cup \{\mathtt{t}\}$; thus $W(\gamma, \mathtt{t}) = \mathtt{T}$ for all $\gamma \in \Gamma$.

First, the basis cases: If $\gamma \equiv \mathtt{T}$ or $\gamma \equiv \mathtt{F}$, the result is immediate. If $\gamma \equiv f$ for $f \in \mathcal{F}$, then

$$\begin{aligned} &W(\gamma, \alpha') = \\ &\quad = \quad W(f, \alpha') \\ &\quad = \quad (\ulcorner[\alpha']f\urcorner \in W) \\ &\quad = \quad I'([\alpha']f) \\ &\quad = \quad I(f, \phi'(\alpha')). \end{aligned}$$

If $\gamma \equiv \alpha R \beta$ for $\alpha, \beta \in \mathcal{T}^*$ and $R \in \{=, \leq, <, \geq, >\}$, then

$$\begin{aligned} &W(\gamma, \alpha') = \\ &\quad = \quad W(\alpha R \beta, \alpha') \\ &\quad = \quad (\ulcorner \alpha R \beta \urcorner \in W) \\ &\quad = \quad I'(\alpha R \beta, \phi'(\alpha')) \end{aligned}$$

Now, the induction: If $\gamma \equiv \neg\delta$, then

$$\begin{aligned} &W(\gamma, \alpha') = \\ &\quad = \quad W(\neg\delta, \alpha') \\ &\quad = \quad \neg W(\delta, \alpha') \\ &\quad = \quad \neg I(\delta, \phi'(\alpha')) \qquad \text{induction} \\ &\quad = \quad I(\neg\delta, \phi'(\alpha')). \end{aligned}$$

If $\gamma \equiv \delta_1 \oplus \delta_2$ for $\oplus \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$, then

$$\begin{aligned}
&W(\gamma, \alpha') = \\
&\quad = \quad W(\delta_1 \oplus \delta_2, \alpha') \\
&\quad = \quad W(\delta_1, \alpha') \oplus W(\delta_2, \alpha') \\
&\quad = \quad I(\delta_1, \phi'(\alpha')) \oplus I(\delta_2, \phi'(\alpha')) \quad \text{induction} \\
&\quad = \quad I(\delta_1 \oplus \delta_2, \phi'(\alpha')).
\end{aligned}$$

If $\gamma \equiv [\alpha]\delta$, then

$$\begin{aligned}
&W(\gamma, \alpha') = \\
&\quad = \quad W([\alpha]\delta, \alpha') \\
&\quad = \quad W(\delta, \alpha) \\
&\quad = \quad I(\delta, \phi'(\alpha)) \qquad \text{induction} \\
&\quad = \quad I([\alpha]\delta, \phi'(\alpha')).
\end{aligned}$$

$\Leftarrow$) Suppose that W is a syntactic Λ-model of Γ, i.e. that W is a satisfiable syntactic Λ-interpretation, and $W(\gamma, \mathtt{t}) = \mathtt{T}$ for all $\gamma \in \Gamma$. Let $I = \langle h, \phi \rangle$ be a model of W (note that W contains only closed formulae). We need to show that $I(\gamma, \phi(\mathtt{t})) = \mathtt{T}$ for all $\gamma \in \Gamma$. By induction we prove the stronger result that $I(\gamma, \phi(\alpha')) = W(\gamma, \alpha')$ for all $\gamma \in \Lambda$ and $\alpha' \in T \cup \{\mathtt{t}\}$.

First, the basis cases: If $\gamma \equiv \mathtt{T}$ or $\gamma \equiv \mathtt{F}$, the result is immediate. If $\gamma \equiv f$ for $f \in \mathcal{F}$, then

$$\begin{aligned}
&I(\gamma, \phi(\alpha')) = \\
&\quad = \quad I(f, \phi(\alpha')) \\
&\quad = \quad I([\alpha']f) \\
&\quad = \quad (\ulcorner [\alpha']f \urcorner \in W) \\
&\quad = \quad W(f, \alpha').
\end{aligned}$$

If $\gamma \equiv \alpha R \beta$ for $\alpha, \beta \in \mathcal{T}^*$ and $R \in \{=, \leq, <, \geq, >\}$, then

$$\begin{aligned}
&I(\gamma, \phi(\alpha')) = \\
&\quad = \quad I(\alpha R \beta, \phi(\alpha')) \\
&\quad = \quad (\alpha R \beta \in W) \\
&\quad = \quad W(\alpha R \beta, \alpha').
\end{aligned}$$

Now, the induction: If $\gamma \equiv \neg\delta$, then

$$\begin{aligned}
&I(\gamma, \phi(\alpha')) = \\
&\quad = \quad I(\neg\delta, \phi(\alpha')) \\
&\quad = \quad \neg I(\delta, \phi(\alpha')) \\
&\quad = \quad \neg W(\delta, \alpha') \\
&\quad = \quad W(\neg\delta, \alpha').
\end{aligned}$$

If $\gamma \equiv \delta_1 \oplus \delta_2$ for $\oplus \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$, then

$$\begin{aligned}
&I(\gamma, \phi(\alpha')) = \\
&\quad = \quad I(\delta_1 \oplus \delta_2, \phi(\alpha')) \\
&\quad = \quad I(\delta_1, \phi(\alpha')) \oplus I(\delta_2, \phi(\alpha')) \\
&\quad = \quad W(\delta_1, \alpha') \oplus W(\delta_2, \alpha') \\
&\quad = \quad W(\delta_1 \oplus \delta_2, \alpha').
\end{aligned}$$

If $\gamma \equiv [\alpha]\delta$, then

$$\begin{aligned}
&I(\gamma, \phi(\alpha')) = \\
&\quad = \quad I([\alpha]\delta, \phi(\alpha') \\
&\quad = \quad I(\delta, \phi(\alpha)) \\
&\quad = \quad W(\delta, \alpha) \\
&\quad = \quad W([\alpha]\delta, \alpha').
\end{aligned}$$

Thus the result follows. $\square$

Theorem 24 Deciding whether a scenario description is satisfiable is NP-complete.
Proof: By Corollary 15, it remains to prove that the problem is in NP. Let $\Upsilon = \langle \sigma, \text{SCD}, \text{OBS} \rangle$ be a scenario description. A few auxiliary definitions will be needed for the proof.

Define the set Δ of formulae by

$$\Delta = \text{OBS} \cup \{\ulcorner \pi \rightarrow [\alpha]\epsilon \urcorner \mid \langle \alpha, \pi, \texttt{Infl}, \epsilon \rangle \in \text{SCD}\},$$

let $T = \mathit{time}(\Delta)$, $F = \mathit{feat}(\Delta)$, $L = \mathit{linrel}(\Delta)$, and define

$$\Delta' = \{\alpha R \beta \mid \alpha, \beta \in T \wedge R \in \{=, <, >\}\},$$

$$A^+ = \{[\alpha]f \mid \alpha \in T \wedge f \in F\} \cup L \cup \Delta',$$

and

$$A' = A^+ \cup \neg A^+.$$

We say that a set $W \subseteq A'$ is a *syntactic scenario interpretation* iff for any $\phi \in A^+$, exactly one of ϕ and $\neg\phi$ is a member of W.

By employing exactly the same method as in Theorem 23, we can define the *truth value* in W of a formula $\gamma \in \Delta \cup \Delta'$, denoted $W(\gamma)$ (no temporal parameter is needed, since all formulae in $\Delta \cup \Delta'$ are closed). Exactly as in Theorem 23, W satisfies the following:

- If I is a model of $\Delta \cup \Delta'$, and we set

$$W = \{\delta \in A' \mid I(\delta) = \mathtt{T}\},$$

then W is a syntactic Λ-model of $\Delta \cup \Delta'$, and I is a model of W.

- If W is a syntactic Λ-model of $\Delta \cup \Delta'$ which is satisfiable by an interpretation I, then I is a model of $\Delta \cup \Delta'$.

Two more auxiliary definitions are needed, for W being a syntactic scenario interpretation.

First, let $\alpha, \beta \in W$. Then we say that α *precedes* β in W, written $\alpha \lhd \beta$, iff $\alpha < \beta \in W$, and for no $\beta' \in T$, $\alpha < \beta' \in W$ and $\beta' < \beta \in W$ holds. If for no α', $\alpha' \lhd \beta$, we write $-\infty \lhd \beta$, and if for no β', $\alpha \lhd \beta'$, we write $\alpha \lhd \infty$.

Second, W is a *syntactic scenario model* of the scenario description Υ iff the following holds:

- W is a syntactic Λ-model of Δ

- For each $f \in \mathcal{F}$ and $\alpha, \beta \in T$ such that $\alpha \lhd \beta$ in W, if there is no action expression $\langle \beta, \pi, \mathtt{Infl}, \epsilon \rangle \in$ SCD with $f \in \mathtt{Infl}$ for some π, $\mathtt{Infl}$ and ϵ, then $[\alpha]f \in W$ iff $[\beta]f \in W$.

Now, the following remains in order to prove the result of the theorem:

- If Υ is satisfiable, then there exists a syntactic scenario model W for Υ

- If there exist a syntactic scenario model W for Υ, then Υ is satisfiable.

This would suffice for proving NP-membership, due to the following facts:

- The size of W is polynomial in Υ

- It can be checked in polynomial time whether W is a syntactic Λ-model of $\Delta \cup \Delta'$ or not by the same argument as in Theorem 23

- Checking the second condition of the definition of syntactic scenario model is polynomial, since we can start by sorting elements of T by the order imposed on T by W (α comes before β iff $\alpha < \beta \in W$), and then proceed with the simple checks of SCD, which is obviously polynomial.

$\Rightarrow$) Suppose that Υ is satisfiable with an intended model $D = \langle h, \phi \rangle$. Now D is a clearly a model of Δ, and we can set

$$W = \{\delta \in A' \mid D(\delta) = \mathtt{T}\}.$$

Clearly W is a syntactic Λ-model of Δ. It remains to check the second condition in the definition of syntactic scenario model. Take $f \in \mathcal{F}$ and $\alpha, \beta \in T$ satisfying the required conditions, and suppose there is no action expression $\langle \beta, \pi, \mathtt{Infl}, \epsilon \rangle \in \textsc{scd}$ such that $f \in \mathtt{Infl}$. By the definition of T and our assumption, we cannot have that $Chg(D, \textsc{scd}, f, t')$ is true for any t' with $D(\alpha) < t' \leq D(\beta)$. Since there are only finitely many points where Chg is true, we can choose a time point $t'' > D(\beta)$ such that for no $s \in [D(\beta), t'')$, $Chg(D, \textsc{scd}, f, s)$ is true. Now, the definition of intended model yields that $[\alpha]f$ is true iff $[\beta]f$ in D, and by the definition of W, that $[\alpha]f \in W$ iff $[\beta]f \in W$.

$\Leftarrow$) Suppose that W is a syntactic scenario model for Υ, which is satisfied by a model $I = \langle h, \phi \rangle$. We construct an intended model $D = \langle h', \phi \rangle$ from I as follows. For each feature $f \in F$ do the following. First for each α such that $-\infty \lhd \alpha$ in W, set $h'(t)(f) = h(\phi(\alpha))(f)$ for every $t \leq \phi(\alpha)$. Then for each α, β with $\alpha \lhd \beta$ in W, set $h'(t)(f) = h(\phi(\alpha))(f)$ for every t such that $\phi(\alpha) \leq t < \phi(\beta)$. Finally, for each α such that $\alpha \lhd \infty$, set $h'(t)(f) = h(\phi(\alpha))(f)$ for every $t \geq \phi(\alpha)$. It is clear that h' is defined for every f and t. It remains to verify that D is an intended model for Υ.

Since W is a syntactic scenario model for Υ, it is also a syntactic Λ-model of Δ. By the fact that I is a model of W, and since D and I agree on the values of all time point expressions in T, D is also a model of W, and by what we know about syntactic Λ-interpretations, D is a model of Δ. Thus, what is left to verify is the second condition in the definition of intended model.

For this purpose, let f be a feature and $s, t \in \mathbb{R}$ with $s < t$, such that for no $t' \in (s, t)$, $Chg(D, \textsc{scd}, f, t')$ holds. We want to prove that $h'(t')(f) = h'(s)(f)$ for every $t' \in (s, t)$. Suppose to the contrary that for some t' with $s < t' < t$, $h'(t')(f) \neq h'(s)(f)$. It is easy to see that by the construction of D from I, this cannot hold, and we have a contradiction. Thus D is an intended model of Υ. $\Box$

4.4 Tractable Scenario Descriptions

Using Corollary 21, we see that if we can code scenario descriptions into sets of Horn formulae, we will have a polynomial algorithm for reasoning with scenario descriptions. In order to obtain such a result, we need to restrict what scenario descriptions are allowed.

The strategy can briefly be described as follows: we identify all observation time points which bind a feature value and all time points where an action expression possibly can change a feature value. Then we connect bound literals with biconditionals, between time points where the literal value should not change. For example, if some action expression changes the value of the feature f at time point α, there exists a $\gamma \in \text{OBS}$ which binds f at a time point β, $\alpha < \beta$, and no changes of the value of f occurs between α and β, then $[\alpha]f \leftrightarrow [\beta]f$ should be added to the theory. This formula can be rewritten in Horn form. The example represents one of the six cases (case 3). The other cases are similar.

The restrictions are basically two: First we will have to represent action expressions as Horn formulae (*restricted* action expressions). Second, the scenario descriptions must be *ordered*; for example, we could not remove the restriction $\alpha < \beta$ in the example above.

Definition 25 Let $\Upsilon = \langle \sigma, \text{SCD}, \text{OBS} \rangle$ be a scenario description. For each $f \in \mathcal{F}$, define

$$E_f = \{\alpha \mid \langle \alpha, \pi, \texttt{Infl}, \epsilon \rangle \in \text{SCD} \wedge f \in \texttt{Infl}\}$$

and

$$C_f = \{\alpha \mid \alpha \textit{ binds } f \textit{ in } \gamma \wedge \gamma \in O\},$$

for

$$O = \text{OBS} \cup \{\pi \mid \langle \alpha', \pi, \texttt{Infl}, \epsilon \rangle \in \text{SCD}\}.$$

E_f is *ordered* iff for $\alpha, \beta \in E_f$, exactly one of $\alpha < \beta$, $\alpha = \beta$ and $\alpha > \beta$ is consistent with[2] OBS. For $\alpha, \beta \in E_f$, E_f ordered, we define $\alpha \prec_f \beta$ iff $\alpha < \beta$ is consistent with OBS and for every $\gamma \in E_f$, $\alpha < \gamma < \beta$ is inconsistent with OBS, $-\infty \prec_f \alpha$ iff for no $\beta \in E_f$, $\beta < \alpha$ is consistent with OBS, and $\alpha \prec_f \infty$ iff for no $\beta \in E_f$, $\alpha < \beta$ is consistent with OBS.

Let $\alpha \in E_f$, $\beta \in \mathcal{T}^*$, and define $\alpha \ll_f \beta$ iff $\alpha \leq \beta$ is consistent with OBS, $\alpha > \beta$ is inconsistent with OBS, and for every $\gamma \in E_f$, $\alpha < \gamma \leq \beta$ is inconsistent with OBS. Also define $-\infty \ll_f \beta$ iff for every $\alpha \in E_f$ $\alpha \leq \beta$ is inconsistent with OBS.

[2] A formula γ is consistent with a set Γ iff $\Gamma \cup \{\gamma\}$ is satisfiable.

If for all $f \in \mathcal{F}$, E_f is ordered, and for all $\omega \in C_f$, $\alpha \ll_f \omega$ for some $\alpha \in E_f \cup \{-\infty\}$, then Υ is *ordered.* The last condition says that for each observation of a feature f, there is a unique change of f which sets its value, or it precedes all changes of f. $\square$

Proposition 26 Testing whether a scenario description Υ is ordered is polynomial, if OBS is Horn.
Proof: For the orderedness of E_f, check for each feature that all pairs of result time points satisfy the condition. Since OBS is Horn, Corollary 21 guarantees that this can be done in polynomial time. The check for $\ll_f$ is polynomial in the same way. $\square$

Definition 27 Let $A = \langle \alpha, \pi, \mathtt{Infl}, \epsilon \rangle$ be an action expression. Then A is *restricted* iff either of the following holds:

- π is $\mathtt{T}$ and ϵ is a conjunction of propositional Horn formulae
- π is a disjunction of negative literals, and ϵ is either $\mathtt{T}$, or a conjunction of negative literals.

An ordered scenario description $\Upsilon = \langle \sigma, \text{SCD}, \text{OBS} \rangle$ is *restricted* iff every action expression in SCD is restricted and OBS is Horn. A restricted scenario description is *normal* iff it is ordered, and for every action expression $A \in \text{SCD}$ either of the following holds:

- π is $\mathtt{T}$ and ϵ is a propositional Horn formula
- π is a negative literal and ϵ is either $\mathtt{T}$ or a negative literal.

$\square$

Of the two examples previously stated, Example 7 is restricted (breaking apart the conjunction of the first observation into separate formulae), whereas Example 8) is not, which is easy to verify (the second one fails to be ordered).

The following result will make the forthcoming proofs easier.

Proposition 28 Let Υ be a restricted scenario description. Then we can construct an equivalent normal scenario description Υ' in polynomial time.
Proof: We note that a restricted action expression

$$A = \langle \alpha, \pi, \mathtt{Infl}, \epsilon \rangle \in \text{SCD}$$

where $\pi \equiv \bigvee_i \neg l_i$ is equivalent to replacing it by one action expression $\langle \alpha, \neg l_i, \mathtt{Infl}, \epsilon \rangle$ for each disjunct in π. Similarly, a restricted action expression $A = \langle \alpha, \pi, \mathtt{Infl}, \bigwedge_i \phi_i \rangle$ can be split into action expressions A_i by $A_i = \langle \alpha, \pi, \mathtt{Infl}, \phi_i \rangle$ (or even easier if $\epsilon \equiv \mathtt{T}$), and obtaining the same set of intended models. The transformation is clearly polynomial.

Orderedness is preserved, since we do not change the order in which features are changed. $\Box$

Thus, we can assume that our restricted scenario descriptions are normal. Next, we define the function Φ which transforms scenario descriptions into sets of Horn formulae.

Definition 29 First let $A = \langle \alpha, \pi, \mathtt{Infl}, \epsilon \rangle$ be a normal action expression. We have three cases for Φ:

- If $\pi \equiv \mathtt{T}$ and $\epsilon \equiv \bigvee_j l_j$ is propositional Horn, then define

$$\Phi(A) = \{\bigvee_j [\alpha] l_j\}$$

- If $\pi \equiv \neg l$ and $\epsilon \equiv \mathtt{T}$, then define

$$\Phi(A) = \emptyset$$

- If $\pi \equiv \neg l$ and $\epsilon \equiv \neg m$, then define

$$\Phi(A) = \{l \vee [\alpha] \neg m\}.$$

The restriction to normal action expressions should be clear; it implies that $\Phi(A)$ is Horn. For a set $\mathcal{S}$ of action expressions, define

$$\Phi(\mathcal{S}) = \bigcup_{A \in \mathcal{S}} \Phi(A).$$

Let $\Upsilon = \langle \sigma, \textsc{scd}, \textsc{obs} \rangle$ be a restricted scenario description with $\sigma = \langle \mathcal{T}, \mathcal{F} \rangle$, and without loss of generality, assume that each feature in $\mathcal{F}$ occurs in SCD or OBS. Also let b be a fresh time point variable (b standing for "beginning"), and for each $f \in \mathcal{F}$, add a new feature f'. A few construction steps (basically corresponding to the possible relations between time points in E_f and C_f) are necessary. We provide the intuitions behind the constructions as we present them.

1. $$\Gamma_1 = \textsc{obs} \cup \Phi(\textsc{scd}) \cup \{b < \alpha \mid \alpha \in E_f \cup C_f, f \in \mathcal{F}\}.$$

   The observations, the transformed action expressions, and an initial time point are added.

2. $$\Gamma_2 = \{\neg[\alpha]f \vee [b]f, [\alpha]f \vee \neg[b]f \mid f \in \mathcal{F}, \alpha \in C_f, -\infty \ll_f \alpha\}.$$

   No action expression influences f before α where it is bound by an observation, therefore f should have the same value at b as at α. Note that the members of Γ_2 each are equivalent to $[\alpha]f \leftrightarrow [b]f$.

3. $$\Gamma_3 = \{\neg[\alpha]f \vee [\beta]f, [\alpha]f \vee \neg[\beta]f \mid f \in \mathcal{F}, \alpha \in E_f, \beta \in C_f, \alpha \ll_f \beta\}.$$

   f is influenced at α and bound by an observation at a later time point β. No actions have effects between α and β, therefore f should have the same value at β as it had at α.

4. $$\Gamma_4 = \{\neg[\alpha]f' \vee [b]f, [\alpha]f' \vee \neg[b]f \mid f \in \mathcal{F}, \alpha \in E_f, -\infty \prec_f \alpha\}.$$

   This case resembles case 2, with the difference that f is influenced at α. Therefore, we introduce a new feature symbol f' which has the same value at α as f has at b. The new symbols will be handled properly below, in case 6.

5. $$\Gamma_5 = \{\neg[\alpha]f \vee [\beta]f', [\alpha]f \vee \neg[\beta]f' \mid f \in \mathcal{F}, \alpha, \beta \in E_f, \alpha \prec_f \beta\}.$$

   This relates to case 3, just as case 4 relates to case 2.

6. Since E_f is ordered, we can form equivalence classes T_f^α of time points for $\alpha \in E_f$ by

   $$T_f^\alpha = \{\beta \in E_f \mid \alpha = \beta \text{ is consistent with } \textsc{obs}\}.$$

   For each $\alpha \in E_f$, define

   $$\mathcal{P}_f^\alpha = \{\pi \mid \langle \beta, \pi, \texttt{Infl}, \epsilon \rangle \in \textsc{scd} \wedge f \in \texttt{Infl} \wedge \beta \in T_f^\alpha\}.$$

   Now set

   $$\Gamma_6 = \{\bigvee \mathcal{P}_f^\alpha \vee \neg[\alpha]f \vee [\alpha]f', \bigvee \mathcal{P}_f^\alpha \vee \neg[\alpha]f' \vee [\alpha]f \mid \alpha \in E_f\}.$$

   First note that this set is equivalent to the set

   $$\{\neg \bigvee \mathcal{P}_f^\alpha \rightarrow ([\alpha]f \leftrightarrow [\alpha]f') \mid \alpha \in E_f\}.$$

Here we ensure that when actions do not have effects, f will have the same value as it had the last time it was changed. This value is held by the feature f'.

Now finally set

$$\Phi(\Upsilon) = \bigcup_i \Gamma_i.$$

It is clear that the transformation performed by Φ is polynomial. □

Theorem 30 Let Υ be a restricted scenario description, and set $\Gamma = C(\Phi(\Upsilon))$. Then Υ is satisfiable iff Γ is satisfiable.
Proof: We start by defining a set Γ' which is satisfiable iff Γ is, by first defining sets Γ'_i for $i \in \{1, \ldots, 6\}$, giving $\Gamma' = \bigcup_{1 \leq i \leq 6} \Gamma'_i$:

1. $\Gamma'_1 = \Gamma_1$
2. $\Gamma'_2 = \{[\alpha]f \leftrightarrow [b]f \mid f \in \mathcal{F}, \alpha \in C_f, -\infty \ll_f \alpha\}$
3. $\Gamma'_3 = \{[\alpha]f \leftrightarrow [\beta]f \mid f \in \mathcal{F}, \alpha \in E_f, \beta \in C_f, \alpha \ll_f \beta\}$
4. $\Gamma'_4 = \{[\alpha]f' \leftrightarrow [b]f \mid f \in \mathcal{F}, \alpha \in E_f, -\infty \prec_f \alpha\}$
5. $\Gamma'_5 = \{[\alpha]f \leftrightarrow [\beta]f' \mid f \in \mathcal{F}, \alpha, \beta \in E_f, \alpha \prec_f \beta\}$
6. $\Gamma'_6 = \{\neg(\bigvee_i \pi_i) \rightarrow ([\alpha]f \leftrightarrow [\alpha]f') \mid \mathtt{T} \notin \{\pi_i\} = \mathcal{P}^{\alpha}_f, \alpha \in E_f\}$.

It is easy to see that each Γ'_i is satisfiable iff the corresponding Γ_i is.
$\Rightarrow$) Suppose Υ is satisfiable by an intended model $D = \langle h, \phi \rangle$. We first construct a new interpretation $D' = \langle h', \phi' \rangle$ as follows.

- Define ϕ' by setting $\phi'(s) = \phi(s)$ for all $s \in \mathcal{T} - \{b\}$, and $\phi'(b) = \min\{\phi(\alpha) \mid \alpha \in T\} - 1$.
- For each feature f, set $h'(t)(f) = h(t)(f)$.
- For each feature f, introduce a new feature f', and define $h'(t)(f')$ so that the formulae in Γ'_4 and Γ'_5 are all true. It is clear that this is possible, since these are essentially *definitions*.

We want to prove that D' is a model of Γ. This is done by checking for every $i \in \{1, \ldots, 6\}$, that D' is a model of Γ'_i.

1. For Γ'_1, all formulae in OBS are true in D', by definition. Furthermore, all formulae in $\Phi(\text{SCD})$ will be true, since these code exactly the conditions required by an intended model, in terms of actions. Furthermore, we have set $\phi'(b)$ to be strictly smaller than the value of every time point expression used, so everything in the last part of Γ'_1 will be true in D'.

2. Γ'_2 says that for each observation or action precondition involving a feature f that is not preceded by any effect on the feature f, the value will be the same as the value at b. Since the value of b in D' is less than every time point expression used, this amounts to strict inertia in each f at all time points before the first change in f. This is clearly satisfied in D', since it inherits this property from the intended model D.

3. Γ'_3 says that for any feature f, observations and action precondition involving f coming directly after changes of f with no change in between, f should retain its value from the change. This is the same as inertia, which is clearly satisfied in any intended model, and is inherited from D.

4. Γ'_4 is satisfied by construction.

5. Γ'_5 is satisfied by construction.

6. Γ'_6 says that if none of the actions which can cause a change in f at α has a true precondition, then f and f' coincide at $\alpha \in E_f$. Since by the definitions of the f' formulae, if $\beta \in E_f$, then f' is always forced to have the value at β that f had just before it changed at β; this just amounts to inertia, which holds for f by construction.

$\Leftarrow$) Suppose Γ is satisfiable by a model $I = \langle h, \phi \rangle$. We construct an intended model $D = \langle h', \phi \rangle$ of Υ from I as follows. Start by removing all features f' for features f. Then for each feature f, do the following. First, for each $\alpha \in E_f$ such that $-\infty \prec_f \alpha$, set $h'(t)(f) = h(\phi(b))(f)$ for each t such that $t < \phi(\alpha)$. Then for each $\alpha, \beta \in E_f$ such that $\alpha \prec_f \beta$, set $h'(t)(f) = h(\phi(\alpha))(f)$ for each t such that $\phi(\alpha) \leq t < \phi(\beta)$. Finally, for each $\alpha \in E_f$ such that $\alpha \prec_f \infty$, set $h'(t)(f) = h(\phi(\alpha))(f)$ for each $t \geq \phi(\alpha)$. It is clear that h' is defined for every t and f, and that by construction, for each f and $\alpha \in E_f$, $h(\phi(\alpha))(f) = h'(\phi(\alpha))(f)$. It remains to verify that D is an intended model of Υ.

First note that D is still a model of $\Gamma'_1 \cup \Gamma'_2 \cup \Gamma'_3$, since the truth of all these formulae is preserved by the transformation from I to D. Thus, D is a model of OBS, since all values used to evaluate the truth of formulae in OBS are identical in I and D, due to D being a model of Γ'_2 and Γ'_3.

For the second condition of the definition of intended model, note that the set

$$\Delta = \{\ulcorner \pi \rightarrow [\alpha]\epsilon \urcorner \mid \langle \alpha, \pi, \mathtt{Infl}, \epsilon \rangle \in \textsc{scd}\}$$

is satisfiable iff $\Phi(\textsc{scd})$ is, by construction, and that this is equivalent to the second condition being satisfied. Thus, since I is a model of $\Phi(\textsc{scd})$, I is also a model of Δ. It remains to check that D is also a model of Δ. Exactly as for the previous condition, the truth values of the π component will not change from I to D. Furthermore, by construction, D will have the same values as I on the $[\alpha]\epsilon$ expressions, since values of features f at time points $\alpha \in E_f$ are the same in D and I. Thus D is also a model of Δ.

Now the proof for the third condition. Suppose that for a feature f and time points $s, t \in \mathbb{R}$ with $s < t$, we have that for no $t' \in (s, t)$, $\mathit{Chg}(D, \textsc{scd}, f, t')$ holds, but $h'(t')(f) \neq h'(s)(f)$ for some $t' \in (s, t)$. Now, it has to hold that $t' = \phi(\alpha')$ for some $\alpha' \in E_f$, by the construction of D.

First suppose $-\infty \prec_f \alpha'$. By the construction of D, we have that $h'(s)(f) = h'(\phi(b))(f)$, and that $h'(\phi(b))(f) = h(\phi(b))(f)$, and thus $h'(s)(f) = h(\phi(b))(f)$. Since I is a model of Γ'_4, $h(\phi(\alpha'))(f') = h'(s)(f)$; so by assumption, $h(\phi(\alpha'))(f') \neq h'(\phi(\alpha'))(f)$, and by the construction of D, it holds that $h(\phi(\alpha'))(f') \neq h(\phi(\alpha'))(f)$. I is a model of Γ'_6; thus $\neg \bigvee \mathcal{P}_f^{\alpha'}$ must be false in I, and so π is true in I for some $\langle \alpha', \pi, \mathtt{Infl}, \epsilon \rangle \in \textsc{scd}$ with $f \in \mathtt{Infl}$. But this means that $\mathit{Chg}(D, \textsc{scd}, f, t')$, a contradiction.

Then suppose $\alpha \prec_f \alpha'$ for some $\alpha \in E_f$. It will suffice to find a contradiction for $s = \phi(\alpha)$, so we make this assumption. Now, by the construction of D, we have that $h'(\phi(\alpha))(f) = h(\phi(\alpha))(f)$, and thus $h'(s)(f) = h(\phi(\alpha))(f)$. Since I is a model of Γ'_5, $h(\phi(\alpha'))(f') = h'(s)(f)$; so by assumption, $h(\phi(\alpha'))(f') \neq h'(\phi(\alpha'))(f)$, and by the construction of D, it holds that $h(\phi(\alpha'))(f') \neq h(\phi(\alpha'))(f)$. I is a model of Γ'_6; thus $\neg \bigvee \mathcal{P}_f^{\alpha'}$ must be false in I, and so π is true in I for some $\langle \alpha', \pi, \mathtt{Infl}, \epsilon \rangle \in \textsc{scd}$ with $f \in \mathtt{Infl}$. But this means that $\mathit{Chg}(D, \textsc{scd}, f, t')$, a contradiction. The result follows. □

Theorem 31 Deciding satisfiability (and entailment) for restricted scenario descriptions is polynomial. □

5 Discussion

One piece of related work is the approach by Schwalb *et al.* [1994] to reasoning about propositions being true at time points. Their choice for obtaining an algorithm is to code both propositions and temporal relations into propositional logic, whereas we do the opposite. However, their tractable inference algorithm is not complete, and they define no measure on when the correct inferences will be obtained, so it is very difficult to relate it to our approach. Furthermore they cannot handle inertia adequately: there, propositions may always change when actions are performed, but certainly this is undesirable if actions which do not affect all features are used.

In this paper our concern has been computational complexity for reasoning about action. It is important to note that although we have provided polynomial algorithms for the reasoning tasks, these can hardly be considered efficient. The important results, however, are that there *exist* polynomial algorithms; the next obvious step is to also make them *fast.* For efficient implementation, there is one direction we are particularly interested in investigating: since the technique used for achieving tractability can be described as an encoding of our logic as temporal constraints for which there is a tractable algorithm for determining satisfiability, it should be possible to do something similar for other tractable temporal algebras, for example those identified by Drakengren and Jonsson [1996, 1997]. Also, an algorithm for a purely *qualitative* scenario description language (i.e. not involving metric time) would probably have a faster satisfiability-checker.

We have shown that satisfiability of scenario descriptions is NP-complete within our formalism. We feel that it would be a mistake to interpret this negatively. On the contrary, one could argue (in lines with [Gottlob, 1996]) that this would imply that many approximations, powerful heuristics and non-trivial tractable subsets of problems for reasoning about action remain to be found. This paper is a step on the way in this endeavour.

6 Conclusions

We have presented a temporal logic and an extension for reasoning about action from which tractable subsets have been extracted. This has been done with an encoding of the logic to Horn DLRs. The formalism is narrative based with continuous time, and the world is modelled with scenario descriptions consisting of action expressions and observations. It is possible to model nondeterminism, concurrency and memory of actions. Time is represented using linear polynomials with rational coefficients over real valued variables.

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Paper VI

A Complete Classification of Tractability in RCC-5

Peter Jonsson and Thomas Drakengren

ABSTRACT

We investigate the computational properties of the spatial algebra RCC-5 which is a restricted version of the RCC framework for spatial reasoning. The satisfiability problem for RCC-5 is known to be NP-complete but not much is known about its approximately four billion subclasses. We provide a complete classification of satisfiability for all these subclasses into polynomial and NP-complete respectively. In the process, we identify all maximal tractable subalgebras which are four in total.

1 Introduction

Qualitative spatial reasoning has received a constantly increasing amount of interest in the literature. The main reason for this is, probably, that spatial reasoning has proved to be applicable to real-world problems in, for example, geographical database systems [Egenhofer, 1991; Grigni *et al.*, 1995] and molecular biology [Cui, 1994]. In both these applications, the size of the problem instances can be huge, so the complexity of problems and algorithms is a highly relevant area to study. However, questions of computational complexity have not received so much attention in the literature; two notable exceptions are the results reported by Nebel [1995] and Renz and Nebel [1997]. In this article we take a small step towards a better understanding of complexity issues in qualitative spatial reasoning.

A well-known framework for qualitative spatial reasoning is the so-called RCC approach [Randell and Cohn, 1989; Randell *et al.*, 1992]. This approach is based on modelling qualitative spatial relations between regions using first-order logic. Of special interest, from a complexity-theoretic standpoint, are the two subclasses RCC-5 and RCC-8. It is well-known that both RCC-5 and RCC-8 have quite weak expressive power. Although they can be used to describe spatial situations, they are very general and should perhaps better be described as topological algebras. However, we will denote these algebras as spatial algebras in order to avoid terminological confusion; the term topological algebra has a well-established but completely different meaning in mathematics [Mallios, 1986].

Bennett [1994] has shown the sufficiency of using propositional logics for reasoning about RCC-5 and RCC-8. Hence, the reasoning becomes more efficient when compared to reasoning in a full first-order logic. Bennett's approach uses classical propositional logic for RCC-5 and intuitionistic propositional logic for RCC-8. Unfortunately, these logics are known to be computationally hard. The satisfiability problem for classical propositional logic and intuitionistic propositional logic is NP-complete [Cook, 1971] and PSPACE-complete [Statman, 1979] respectively. However, the complexity of the underlying logic does not carry over in both cases; Renz and Nebel [1997] have shown that the satisfiability problem for both RCC-5 and RCC-8 is NP-complete. The full proofs can be found in [Renz, 1996].

These findings motivate the search for tractable subclasses of RCC-5 and RCC-8. Nebel [1995] showed that reasoning with the basic relations in RCC-8 is a polynomial-time problem. Renz and Nebel [1997] improved this result substantially by showing the following results:

- There exists a large, maximal subclass of RCC-8, denoted $\widehat{\mathcal{H}}_8$, which contains all basic relations and is polynomial. $\widehat{\mathcal{H}}_8$ contains 148 elements out of 256 (58%).

- There exists a large, maximal subclass of RCC-5, denoted $\widehat{\mathcal{H}}_5$, which contains all basic relations and is polynomial. $\widehat{\mathcal{H}}_5$ contains 28 elements out of 32 (87%). Furthermore, this is the unique, maximal subclass of RCC-5 containing all basic relations.

We will concentrate on RCC-5 in this article. The main result is a complete classification of all subclasses of RCC-5 with respect to tractability. The classification makes it possible to determine whether a given subclass is tractable or not by a simple test that can be carried out by hand or automatically. We have thus gained a clear picture of the tractability borderline in RCC-5. As is more or less necessary when showing results of this kind, the main proof relies on a case analysis performed by a computer. The number of cases considered was roughly 4×10^4. The analysis cannot, of course, be reproduced in a research paper or be verified manually. Hence, we include a description of the programs used. The programs are also available as an on-line appendix to this article.

The structure of the article is as follows: Section 2 defines RCC-5 and some auxiliary concepts. Section 3 contains the tractability proofs for three subclasses of RCC-5. In Section 4 we show that these subclasses together with $\widehat{\mathcal{H}}_5$ are the only maximal tractable subclasses of RCC-5. The article concludes with a brief discussion of the results.

2 The RCC-5 Algebra

We follow Bennett [1994] in our definition of RCC-5. RCC-5 is based on the notions of *regions* and *binary relations* on them. A region p is a variable interpreted over the non-empty subsets of some fixed set. It should be noted that we do not require the sets to be open sets in some topological space. This is no limitation since it is impossible to distinguish interior points from boundary points in RCC-5. Thus we can take any set $\mathcal{X}$ and use the discrete topology $\mathcal{T} = \langle \mathcal{X}, 2^{\mathcal{X}} \rangle$, where every subset of $\mathcal{X}$ is an open set in $\mathcal{T}$.

We assume that we have a fixed universe of variable names for regions. Then, an *R-interpretation* is a function that maps region variables to the non-empty subsets of some set.

Given two interpreted regions, their relation can be described by exactly one of the elements of the set $\mathbf{B}$ of five *basic RCC-5 relations*. The definition of these relations can be found in Table 1. Figure 1 shows 2-dimensional examples of the relations in RCC-5. A formula of the form XBY where X and Y are regions and $B \in \mathbf{B}$, is said to be satisfied by an R-interpretation iff the interpretation of the regions satisfies the relations specified in Table 1.

To express indefinite information, unions of the basic relations are used, written as sets of basic relations, leading to 2^5 binary RCC-5 relations. Naturally, a set of basic relations is to be interpreted as a disjunction of the basic relations. The set of all RCC-5 relations $2^{\mathbf{B}}$ is denoted by R_5. Relations of special interest are the *null* relation $\varnothing$ (also denoted by $\perp$) and the *universal* relation $\mathbf{B}$ (also denoted $\top$).

A formula of the form $X\{B_1, \ldots, B_n\}Y$ is called an RCC-5 formula. Such a formula is satisfied by an R-interpretation $\Im$ iff XB_iY is satisfied by $\Im$ for some i, $1 \leq i \leq n$. A finite set Θ of RCC-5 formulae is said to be *R-satisfiable* iff there exists an R-interpretation $\Im$ that satisfies every formula of Θ. Such a satisfying R-interpretation is called an *R-model* of Θ. Given an R-interpretation $\Im$ and a variable v, we write $\Im(v)$ to denote the value of v under the interpretation $\Im$.

The reasoning problem we will study is the following:

INSTANCE: A finite set Θ of RCC-5 formulae.
QUESTION: Does there exist an R-model of Θ?

We denote this problem by RSAT. In the following, we often consider restricted versions of RSAT where the relations used in formulae in Θ are only from a subset S of R_5. In this case we say that Θ is a set of formulae over S and we use a parameter in the problem description to denote the subclass under consideration, *e.g.*, RSAT(S). Note that an RSAT problem instance can be represented by a labelled directed graph, where the nodes are region variables and the arcs are labelled by relations between variables. Given an instance Θ of RSAT, we say that such a graph is a *graph representation* of Θ.

We continue by defining an algebra over the RCC-5 relations.

Definition 2.1 Let $\mathbf{B} = \{\mathtt{DR}, \mathtt{PO}, \mathtt{PP}, \mathtt{PPI}, \mathtt{EQ}\}$. The RCC-5 algebra consists of the set $R_5 = 2^{\mathbf{B}}$ and the operations unary *converse* (denoted by $\smile$), binary *intersection* (denoted by $\cap$) and binary *composition* (denoted by $\circ$). They are defined as follows:

$X\{\mathtt{DR}\}Y$	iff	$X \cap Y = \varnothing$
$X\{\mathtt{PO}\}Y$	iff	$\exists a, b, c : a \in X, a \notin Y, b \in X, b \in Y, c \notin X, c \in Y$
$X\{\mathtt{PP}\}Y$	iff	$X \subset Y$
$X\{\mathtt{PPI}\}Y$	iff	$X \supset Y$
$X\{\mathtt{EQ}\}Y$	iff	$X = Y$

Table 1: The five basic relations of RCC-5.

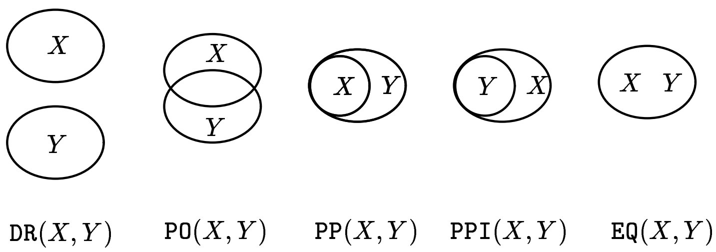

Figure 1: Pictorial example of the relations in RCC-5.

$$\begin{array}{lrll} \forall X,Y: & XR^{\smile}Y & \text{iff} & YRX \\ \forall X,Y: & X(R\cap S)Y & \text{iff} & XRY \wedge XSY \\ \forall X,Y: & X(R\circ S)Y & \text{iff} & \exists Z : (XRZ \wedge ZSY) \end{array}$$

If $\mathcal{S}$ is a subset of R_5, $\mathcal{S}$ is said to be a *subalgebra* of RCC-5 iff $\mathcal{S}$ is closed under converse, intersection and composition. It can easily be verified that $R \circ S = \bigcup\{B \circ B' | B \in R, B' \in S\}$, *i.e.*, composition is the union of the component-wise composition of basic relations.

Next, we introduce a *closure* operation. The closure operation transforms a given subclass of R_5 to one that is polynomially equivalent to the original subclass with respect to satisfiability. The operation is similar to the closure operation for RCC-5 introduced by Renz [1996] but it does not pose the same restrictions on the given subclass. (Renz's operation requires $\{\mathtt{EQ}\}$ to be a member of the subclass to be closed.)

Definition 2.2 Let $\mathcal{S} \subseteq R_5$. Then we denote by $\overline{\mathcal{S}}$ the *closure* of $\mathcal{S}$, defined as the least subalgebra containing $\mathcal{S}$ closed under converse, intersection and composition.

Observe that a subset $\mathcal{S}$ of R_5 is a subalgebra iff $\mathcal{S} = \overline{\mathcal{S}}$.

The next lemma is given without proof. A proof of the analogous result for Allen's algebra can be found in Nebel and Bürckert [1995].

Lemma 2.3 Let $\mathcal{S} \subseteq R_5$. Then RSAT($\overline{\mathcal{S}}$) can be polynomially transformed to RSAT($\mathcal{S}$) and vice versa.

Corollary 2.4 Let $\mathcal{S} \subseteq R_5$. RSAT($\mathcal{S}$) is polynomial iff RSAT($\overline{\mathcal{S}}$) is polynomial. RSAT($\mathcal{S}$) is NP-complete iff RSAT($\overline{\mathcal{S}}$) is NP-complete.

3 Tractable Subclasses of RCC-5

We begin this section by defining four tractable subalgebras of RCC-5, which can be found in Table 1. Later on, we show that these algebras are the only maximal tractable subalgebras of RCC-5. The tractability of the first algebra, R_5^{28}, has been established by Renz and Nebel [1997]. The name R_5^{28} reflects the fact that the algebra contains 28 elements.

Theorem 3.1 RSAT(R_5^{28}) is polynomial.

The tractability of our second algebra, R_5^{20}, can be settled quite easily. The algorithm can be found in Figure 2.

Lemma 3.2 Let Θ be an instance of RSAT(R_5^{20}). The algorithm A^{20} accepts on input Θ iff Θ has an R-model.

Proof: *if:* We show the contrapositive, *i.e.*, if A^{20} rejects then Θ has no R-model. Clearly, the satisfiability of Θ is preserved under the transformations made in lines 7-10. Note that if $XRX \in \Theta$ then $\texttt{EQ} \in R$ if Θ is satisfiable. Thus Θ is not satisfiable if the algorithm rejects in line 5. Similarly, Θ is not satisfiable if the algorithm rejects in line 6.

only-if: Consider the set Θ after the completion of line 11. We denote this set by Θ'. Obviously, Θ' is satisfiable if the initial Θ was satisfiable. Also observe that line 7 ensures that Θ' does not relate any variables with `EQ`. Furthermore, line 8 guarantees that there is at most one relation that relates two variables.

Now, we construct an R-model M for Θ' as follows: Let V be the set of variables in Θ'. Let M assign non-empty sets that are pairwise disjoint to the members of V. Let $\mathcal{U} = \bigcup_{X \in V} M(X)$. Introduce a set of values $U' = \{\alpha_{X,Y} \mid X, Y \in V\}$ satisfying the following:

	R_5^{28}	R_5^{20}	R_5^{17}	R_5^{14}
$\perp$	•	•	•	•
{DR}	•	•		
{PO}	•	•		
{DR,PO}	•	•		
{PP}	•			•
{DR,PP}	•	•		
{PO,PP}	•			
{DR,PO,PP}	•	•		
{PPI}	•			•
{DR,PPI}	•	•		
{PO,PPI}	•			
{DR,PO,PPI}	•	•		
{PP,PPI}				•
{DR,PP,PPI}		•		•
{PO,PP,PPI}	•			•
{DR,PO,PP,PPI}	•	•		•
{EQ}	•	•	•	•
{DR,EQ}	•	•	•	
{PO,EQ}	•	•	•	
{DR,PO,EQ}	•	•	•	
{PP,EQ}	•		•	•
{DR,PP,EQ}	•	•	•	
{PO,PP,EQ}	•		•	
{DR,PO,PP,EQ}	•	•	•	
{PPI,EQ}	•		•	•
{DR,PPI,EQ}	•	•	•	
{PO,PPI,EQ}	•		•	
{DR,PO,PPI,EQ}	•	•	•	
{PP,PPI,EQ}			•	•
{DR,PP,PPI,EQ}		•	•	•
{PO,PP,PPI,EQ}	•		•	•
$\top$	•	•	•	•

Table 1: The maximal tractable subalgebras of RCC-5.

1. $\alpha_{X,Y} = \alpha_{Z,W}$ iff $X = Z$ and $Y = W$; and

2. for arbitrary $X, Y \in V$, $\alpha_{X,Y} \notin \mathcal{U}$.

For each relation of the type $X\{\texttt{PO}\}Y$ or $X\{\texttt{PO},\texttt{EQ}\}Y$, extend the sets $M(X)$ and $M(Y)$ with the element $\alpha_{X,Y}$.

Clearly, two sets X, Y are disjoint (and are thus related by DR) under M unless $X\{\texttt{PO}\}Y$ or $X\{\texttt{PO},\texttt{EQ}\}Y$ is in Θ. But in these cases, X and Y must not be disjoint. In fact, by introducing $\alpha_{X,Y}$, we have forced $X\{\texttt{PO}\}Y$ to hold under M which satisfies formulae of the type $X\{\texttt{PO}\}Y$ as well as formulae of the type $X\{\texttt{PO},\texttt{EQ}\}Y$. Hence, M is an R-model of Θ' which implies the R-satisfiability of Θ. □

Theorem 3.3 RSAT(R_5^{20}) is polynomial.

Proof: Algorithm A^{20} correctly solves the RSAT(R_5^{20}) problem by the previous lemma. Furthermore, the number of iterations is bounded from above by the number of variables and the number of formulae in the given instance and the tests can easily be performed in polynomial time. □

Next we show the tractability of RSAT(R_5^{17}).

Theorem 3.4 RSAT(R_5^{17}) is polynomial.

Proof: Consider the algorithm A^{17} in Figure 2. If there exist X, Y such that $X \bot Y \in \Theta$ then Θ is not satisfiable. Otherwise, we can let all variables have the same value. Since EQ is a member of every relation that occurs in Θ, this interpretation is an R-model of Θ. □

We continue by proving that RSAT(R_5^{14}) is a tractable problem. Let

$$R_5^9 = \{\{\texttt{PP},\texttt{EQ}\}\} \cup \{R \cup \{\texttt{PP},\texttt{PPI}\} \mid R \in R_5\}.$$

Using a machine-assisted proof, it can be shown that $R_5^{14} = \overline{R_5^9}$ so it is sufficient to prove the tractability of RSAT(R_5^9) by Corollary 2.4. The program that we used for showing this is available as an on-line appendix to this article.

From now on, let Θ be an arbitrary instance of RSAT(R_5^9) and $G = \langle V, E\rangle$ be its graph representation. The following proofs are similar in spirit to some of the proofs appearing in Drakengren and Jonsson [1996]. The algorithm itself is reminiscent of an algorithm by van Beek [1992] for deciding satisfiability in the point algebra.

algorithm A^{20}
Input: An instance Θ of RSAT(R_5^{20}).
repeat
  $\Theta' \leftarrow \Theta$
  if $\exists X, R : XRX \in \Theta$ and $\texttt{EQ} \notin R$ **then reject**
  if $\exists X, Y : X \perp Y \in \Theta$ **then reject**
  if $\exists X, Y : X \neq Y$ and $X\{\texttt{EQ}\}Y \in \Theta$ **then** substitute Y for X in Θ
  if $\exists X, Y, R, S : XRY \in \Theta$ and $XSY \in \Theta$ **then**
    $\Theta \leftarrow (\Theta - \{XRY, XSY\}) \cup \{X(R \cap S)Y\}$
  if $\exists X, R : XRX \in \Theta$ and $\texttt{EQ} \in R$ **then** $\Theta \leftarrow \Theta - \{XRX\}$
until $\Theta = \Theta'$
accept

algorithm A^{17}
Input: An instance Θ of RSAT(R_5^{17}).
if $\exists X, Y$ such that $X \perp Y \in \Theta$ **then reject**
else accept

algorithm A^{9}
Input: An instance Θ of RSAT(R_5^{9}) with graph representation G.
Let G' be the graph obtained from G by removing arcs which are not labelled $\{\texttt{PP}, \texttt{EQ}\}$.
Find all strongly connected components C in G'
for every arc e in G whose relation does not contain $\texttt{EQ}$ **do**
  if e connects two nodes in some C **then reject**
accept

Figure 2: Algorithms for RSAT(R_5^{20}), RSAT(R_5^{17}) and RSAT(R_5^{9}).

Definition 3.5 A RCC-5 relation R is said to be an *acyclic* relation iff any cycle in any G with R on every arc is never satisfiable.

The relation PP is an example of an acyclic relation while $\{\texttt{PP}, \texttt{EQ}\}$ is not acyclic. We continue by showing a few properties of acyclic relations.

Proposition 3.6 Let R be an acyclic relation. Then every relation $R' \subseteq R$ is acyclic.

Proof: Since taking subsets of R constrains satisfiability further, the result follows. $\square$

Proposition 3.7 Let R be an acyclic relation, and choose A such that $A \subseteq \{R' \,|\, R' \subseteq R\}$. Then, any cycle in G where every arc is labelled by some relation in A is unsatisfiable.

Proof: Same argument as in the previous proposition. $\square$

The following definition is needed in the following proofs.

Definition 3.8 Let $\mathcal{I}$ be an instance of the R-satisfiability problem, M a model for $\mathcal{I}$, and $r \in R_5$ a relation between two region variables X and Y in $\mathcal{I}$. Then r is said to be *satisfied as* r' in M for any relation $r' \subseteq r$, such that $Xr'Y$ is satisfied in M.

The definition may seem a bit cumbersome but the essence should be clear. As an example, let X and Y be region variables related by $X\{\mathtt{PO},\mathtt{PP}\}Y$, and M a model where X is interpreted as $\{1,2\}$ and Y as $\{1,2,3\}$. Then in M, $\{\mathtt{PO},\mathtt{PP}\}$ is satisfied as $\{\mathtt{PP}\}$, but also as $\{\mathtt{PO},\mathtt{PP}\}$.

Lemma 3.9 Let R be an acyclic relation, and A, A' sets such that $A \subseteq \{R' \mid R' \subseteq R\}$ and $A' \subseteq \{a \cup \{\mathtt{EQ}\} \mid a \in A\}$. Then, every cycle C labelled by relations in $A \cup A'$ is satisfiable iff it contains only relations from A'. Furthermore, all relations in the cycle have to be satisfied as $\mathtt{EQ}$.

Proof: *only-if:* Suppose that a cycle C is satisfiable and that it contains some relation from A. Apply induction on the number n of arcs in the cycle. For $n = 1$, we get a contradiction by Proposition 3.7. So, suppose for the induction that C contains $n+1$ arcs. Let M be an R-model for the relations in C. It cannot be the case that every relation in C is satisfied in M as some relation in A, by Proposition 3.7. Thus, some relation R' in C has to be satisfied as $\mathtt{EQ}$. But then we can collapse the two variables connected by R' to one variable, and we have a cycle with n nodes containing a relation from A. This contradicts the induction hypothesis.

if: Suppose that a cycle C contains only relations in A'. Then C can be satisfied by choosing $\mathtt{EQ}$ on every arc. Notice that the only-if part implies that C *must* be satisfied by choosing $\mathtt{EQ}$ on every arc. Hence, the variables are forced to be equal. $\square$

After having studied acyclic relations, we will now turn our attention to *DAG-satisfying* relations. The formal definition is as follows.

Definition 3.10 A basic relation B is said to be *DAG-satisfying* iff any DAG (directed acyclic graph) labelled only by relations containing B is satisfiable, *i.e.*, if the corresponding RSAT problem has a model.

A typical example of a DAG-satisfying relation is `EQ`. Given a DAG labelled only by relations containing `EQ`, we can always satisfy these relations by assigning some non-empty set S to all variables.

We can now show that `PP` is a DAG-satisfying relation.

Definition 3.11 Let G be an arbitrary DAG. A node v in G is said to be a *terminal* node iff there are no arcs which start in v.

Lemma 3.12 The basic relation `PP` is DAG-satisfying.

Proof: Let G be a DAG labelled only by relations containing `PP`. We show that G is satisfied by some R-model M. Induction on n which is the number of nodes in G. The case when $n = 1$ is trivial. Suppose that G has $n + 1$ nodes and remove a terminal node g. By induction, the remaining graph $G' = \langle V', E' \rangle$ is satisfiable by a model M'. We shall now construct a model M of G, which agrees with M' on every variable in G'. Let $S = \bigcup\{M'(v) \mid v \in V'\}$ and let α be an element not in S. Let $M(g) = S \cup \{\alpha\}$. Obviously, M is a model of G. □

We now state a simple result from Drakengren and Jonsson [1996].

Lemma 3.13 Let G be irreflexive[1] and have an acyclic subgraph D. Then those arcs of G which are not in D can be reoriented so that the resulting graph is acyclic.

By specializing this result, we get the next lemma.

Lemma 3.14 Let G be irreflexive with an acyclic subgraph D and let the arcs of D be labelled by relations containing `PP`, and the arcs not in D being labelled by relations containing `PP` and `PPI`. Then G is R-satisfiable.

[1] A graph is irreflexive iff it has no arcs from a node v to the node v.

Proof: Reorient the arcs of G such that the resulting graph is acyclic. This is always possible by the previous lemma. Furthermore, whenever an arc is reoriented, also invert the relation on that arc, so that G' is satisfiable iff G is. By this construction, only arcs containing both PP and PPI have been reoriented, so every arc in the DAG G' contains PP and, thus, since PP is DAG-satisfying by Lemma 3.12, G' is satisfiable. Consequently, G is also satisfiable. □

Lemma 3.15 Algorithm A^9 correctly solves RSAT(R_5^9).

Proof: Assume that the algorithm finds a strongly connected component of G' (which then contains only the relation $\{\texttt{PP}, \texttt{EQ}\}$), containing two nodes that in G are connected by an arc e that is labelled by a relation R' which does not contain EQ. Then there exists a cycle C in which the relation of every arc contains EQ, such that e connects two nodes in that C but e is not part of that cycle. By Lemma 3.9, C can be satisfied only by choosing the relation EQ on every arc in C, and since R' does not admit EQ, C is unsatisfiable.

Otherwise, every such strongly connected component can be collapsed to a single node, removing all arcs which start and end in the collapsed node. This transformation retains satisfiability using the same argument as above. After collapsing, the subgraph obtained by considering only arcs labelled $\{\texttt{PP}, \texttt{EQ}\}$ will be acyclic. Since the remaining arcs are labelled by relations containing both PP and PPI, the graph is R-satisfiable by Lemma 3.14. (Note that the graph will be irreflexive since every node is contained in some strongly connected component.) □

Lemma 3.16 Given a graph $G = \langle V, E \rangle$, algorithm A^9 runs in $O(|V| + |E|)$ time.

Proof: Strongly connected components can be found in $O(|V| + |E|)$ time [Baase, 1988] and the remaining test can also be made in $O(|V| + |E|)$ time. □

Theorem 3.17 RSAT(R_5^{14}) can be solved in polynomial time.

Proof: RSAT(R_5^9) is polynomial by the previous two lemmata. Since $R_5^{14} = \overline{R_5^9}$, RSAT($R_5^{14}$) can be solved in polynomial time by Corollary 2.4. □

4 Classification of RCC-5

Before we can give the classification of RCC-5 we need two NP-completeness results.

Theorem 4.1 RSAT($\mathcal{S}$) is NP-complete if

1. $\mathcal{C}_1 = \{\{\texttt{PO}\}, \{\texttt{PP}, \texttt{PPI}\}\} \subseteq \mathcal{S}$ [Renz, 1996], or
2. $\mathcal{C}_2 = \{\{\texttt{DR}, \texttt{PO}\}, \{\texttt{PP}, \texttt{PPI}\}\} \subseteq \mathcal{S}$.

Proof: The proof for $\mathcal{C}_2$ is by polynomial-time reduction from RSAT($\mathcal{C}_1$). Let Θ be an arbitrary instance of RSAT($\mathcal{C}_1$). Construct the following set:

$$\Theta' = \{X\{\texttt{PP}, \texttt{PPI}\}Y \mid X\{\texttt{PP}, \texttt{PPI}\}Y \in \Theta\} \cup \{X\{\texttt{DR}, \texttt{PO}\}Y \mid X\{\texttt{PO}\}Y \in \Theta\}.$$

Clearly, Θ' can be obtained from Θ in polynomial time and Θ' is an instance of RSAT($\mathcal{C}_2$). We show that Θ is satisfiable iff Θ' is satisfiable.

only-if: Assume that there exists an R-model I of Θ. It is not hard to see that I is also an R-model of I' since if $X\{\texttt{PO}\}Y$ under I then $X\{\texttt{DR}, \texttt{PO}\}Y$ under I. Thus Θ' is R-satisfiable if Θ is R-satisfiable.

if: Assume the existence of an R-model I' that assigns subsets of some set $\mathcal{U}$ to the region variables of Θ'. Let α be an element such that $\alpha \notin \mathcal{U}$. We construct a new interpretation I as follows: $I(x) = I'(x) \cup \{\alpha\}$ for every variable x in Θ'. It can easily be seen that the following holds for I:

1. If $x\{\texttt{DR}\}y$ under I' then $x\{\texttt{PO}\}y$ under I.
2. If $x\{\texttt{PO}\}y$ under I' then $x\{\texttt{PO}\}y$ under I.
3. If $x\{\texttt{PP}\}y$ under I' then $x\{\texttt{PP}\}y$ under I.
4. If $x\{\texttt{PPI}\}y$ under I' then $x\{\texttt{PPI}\}y$ under I.

It is easy to see that if $x\{\texttt{PP}, \texttt{PPI}\}y$ under I' then $x\{\texttt{PP}, \texttt{PPI}\}y$ under I. Similarly, if $x\{\texttt{DR}, \texttt{PO}\}y$ under I' then $x\{\texttt{PO}\}y$ under I. It follows that I is a model of Θ so Θ is R-satisfiable if Θ' is R-satisfiable. $\square$

The main theorem can now be stated and proved.

Theorem 4.2 For $\mathcal{S} \subseteq R_5$, RSAT($\mathcal{S}$) is polynomial iff $\mathcal{S}$ is a subset of some member of $R_P = \{R_5^{28}, R_5^{20}, R_5^{17}, R_5^{14}\}$, and NP-complete otherwise.

Proof: *if:* For each $R \in R_P$, RSAT(R) is polynomial as was shown in the previous section.
only-if: Choose $\mathcal{S} \subseteq R_5$ such that $\mathcal{S}$ is not a subset of any algebra in R_P. For each subalgebra $R \in R_P$, choose a relation x such that $x \in \mathcal{S}$ and $x \notin R$. This can always be done since $\mathcal{S} \not\subseteq R$. Let X be the set of these relations and note that X is not a subset of any algebra in R_P. The set R_P contains four algebras so by the construction of X, $|X| \leq 4$. Observe that RSAT($\mathcal{S}$) is NP-complete if RSAT(X) is NP-complete.

To show that RSAT($\mathcal{S}$) has to be NP-complete, a machine-assisted case analysis of the following form was performed:

1. Generate all subsets of R_5 of size ≤ 4. There are $\sum_{i=0}^{4} \binom{32}{i} = 41449$ such subsets.
2. Let $\mathcal{T}$ be such a set. Test: $\mathcal{T}$ is a subset of some subalgebra in R_P or $\mathcal{C}_i \subseteq \overline{\mathcal{T}}$ for some $i \in \{1, 2\}$.

The test succeeds for all $\mathcal{T}$. Hence, RSAT($\mathcal{S}$) is NP-complete by Corollary 2.4. □

The program used for showing the previous theorem appears in the on-line appendix of this article.

5 Discussion

The main problem of reporting tractability results for restricted classes of problems is the difficulty of isolating interesting and relevant subclasses. The systematic approach of building complete classifications is a way of partially overcoming this problem. If the problem class under consideration is regarded relevant, then its tractable subclasses should be regarded relevant if the computational problem is of interest. This is especially true in spatial reasoning where the size of the problem instances can be extremely large; one good example is spatial reasoning in connection with the Human Genome project [Cui, 1994].

Another advantage with complete classifications is that they are more satisfactory from a scientific point of view; to gain a clear picture of the borderline between tractability and intractability has an intrinsic scientific value. A common critique is that complete classifications tend to generate certain classes which are totally useless. For instance, the subalgebra R_5^{17} is certainly of no use. It must be made clear that such critique is unjustified since the researcher who makes a complete classification does not deliberately invent useless classes. Instead, if useless classes appear in a complete classification, they are unavoidable parts of the classification.

The work reported in this article can be extended in many different ways. One obvious extension is to study other computational problems than the RSAT problem. Renz [1996] has studied two problems, RMIN and RENT, on certain subclasses of RCC-5 and RCC-8. The RMIN problem is to decide if a set of spatial formulae Θ is minimal, *i.e.*, whether all basic relations in every formula of Θ can be satisfied or not. The RENT problem is to decide whether a formula XRY is entailed by a set of spatial formulae. Grigni *et al.* [1995] study a stronger form of satisfiability which they refer to as *realizability*: A finite set Θ of RCC-5 formulae is said to be realizable iff there exist regions on the plane bounded by Jordan curves which satisfy the relations in Θ. Grigni *et al.* [1995] have shown that the realizability problem is much harder than the satisfiability problem. For instance, deciding realizability of formulae constructed from the two relations `DR` and `PO` is NP-complete while the satisfiability problem is polynomial. Certainly, further studies of the realizability problem would be worthwhile.

Another obvious research direction is to completely classify other spatial algebras, such as RCC-8. RCC-8 contains $2^{256} \approx 10^{77}$ relations so the question whether this is feasible or not remains to be answered.

6 Conclusions

We have studied computational properties of RCC-5. All of the 2^{32} possible subclasses are classified with respect to whether their corresponding satisfiability problem is tractable or not. The classification reveals that there are four maximal tractable subclasses of the algebra.

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Paper VII

Qualitative Reasoning About Sets Applied to Spatial Reasoning

Peter Jonsson and Thomas Drakengren

ABSTRACT

We propose a polynomial-time algorithm for a restricted form of qualitative reasoning about sets. The algorithm decides satisfiability of set expressions of the type XrY where r is one of the three relations of subset, disjointness and non-equality. The algorithm also handles certain disjunctions of such expressions. With the aid of this algorithm, we construct a straightforward, tractable algorithm for the Horn fragment of the spatial theory RCC-5. The algorithm also enables us to show certain topological properties of RCC-5. For instance, we show that a result by Grigni *et al* about RCC-5 restricted to two-dimensional space does not hold for RCC-5 restricted to three-dimensional space.

1 Introduction

Reasoning about sets has proved itself useful in a number of areas in computer science. The typical AI examples are the so-called "terminological languages" [Brachman and Schmolze, 1985]. Outside the AI community, reasoning about sets has been employed in, for example, analyses of programs, logic programming and the study of finite automata. The main result of this paper is a tractable algorithm for deciding satisfiability of a class of set expressions. More specifically, we relate set variables with subset, disjointness and non-equality and we allow certain disjunctions of such set relations.

Even though the problem that we study appears to be severely restricted, it seems to have practical relevance. To demonstrate this, we show how it can be used in qualitative spatial reasoning. This kind of spatial reasoning has received much attention in the literature recently. The main reason for this is, probably, that spatial reasoning has proved to be applicable to real-world problems in, for example, geographical databases [Egenhofer, 1991; Grigni *et al.*, 1995] and molecular biology [Cui, 1994]. In both these applications, the size of the problem instances can be huge, so it is highly relevant to study the complexity of problems and algorithms.

A widespread framework for qualitative spatial reasoning is the RCC approach [Randell and Cohn, 1989; Randell *et al.*, 1992]. This approach is based on modelling qualitative spatial relations between regions using first-order logic. Of special interest is a subtheory of RCC, RCC-5. Bennett [1994] has shown that this subtheory has a much lower complexity figure than the full RCC framework but, unfortunately, the satisfiability problem for RCC-5 is still NP-complete [Renz, 1996].

This motivates the search for tractable subclasses of RCC-5. Renz [1996] has shown that there exists a large, maximal tractable subclass of RCC-5, denoted $\widehat{\mathcal{H}}_5$. $\widehat{\mathcal{H}}_5$ contains 28 elements out of 32 (87%). Renz's proof of the tractability of $\widehat{\mathcal{H}}_5$ relies on a complicated reduction in two steps: the problem is first reduced to a fragment of the modal logic S5 and, second, this modified problem is reduced to propositional Horn logic. We provide an alternative method for $\widehat{\mathcal{H}}_5$. We employ a simple reduction from the satisfiability problem for $\widehat{\mathcal{H}}_5$ to our tractable class of set expressions. This provides two advantages. Firstly, the resulting algorithm is very straightforward and does not rely on a translation of the problem to logic. Secondly, we can use the algorithm for showing topological properties of RCC-5. For example, we complement the findings in Grigni *et al.* [1995] on spatial reasoning with strong topological assumptions. They have shown that the satisfiability problem of a certain subset of RCC-5 is NP-complete if we require the sets to be connected sets in two-dimensional Euclidean space. However, we show that an extended version of this problem is tractable if the allowed sets are connected sets in *three*-dimensional Euclidean space.

In Sections 2 and 3 we introduce the algorithm for reasoning about sets and show that it is tractable. RCC-5 is introduced in Section 4 together with the new tractability proof of $\widehat{\mathcal{H}}_5$. The topological results are collected in Section 5. Finally, the last two sections discuss the results and conclude the paper.

2 Disjunctive Set Relations

Our approach to reasoning about sets is based on *set variables* and *relations* on these. We assume that we have a fixed universe of variable names for sets. Then, an *S-interpretation* is a function that maps set variables to non-empty sets. The reason for restricting ourselves to only consider non-empty sets is that we intend to use the algorithms for spatial reasoning. Hence, we want to disallow the pathological case of having empty objects. It may, however, be necessary to allow empty sets in other applications but this is left for future research.

We will relate set variables by the following three relations: subset ($\subseteq$), non-equality ($\neq$) and disjointness (disj). The relations $\subseteq$ and $\neq$ have their usual definitions and $X \mathsf{disj}\, Y$ holds iff $X \cap Y = \emptyset$.

Two set variables related by one of these three relations is called an *atomic relation.* A *disjunctive set relation* (DSR) is a set of atomic relations.

An atomic relation XRY is satisfied by an S-interpretation I iff the relation $I(X)RI(Y)$ holds. A finite set $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ of DSRs is said to be S-satisfiable iff there exists an S-interpretation that satisfies at least one member of each γ_i, $1 \leq i \leq n$. Such a satisfying S-interpretation is called an S-model of Γ. To define tractable subclasses of these problems, we need to name some special classes of DSRs.

Definition 2.1 Let γ be a DSR. Then we let $\mathcal{S}(\gamma)$ denote the set of subset relations in γ, $\mathcal{NE}(\gamma)$ the set of non-equality relations, $\mathcal{D}(\gamma)$ the set of disjointness relations and $\mathcal{NS}(\gamma) = \mathcal{NE}(\gamma) \cup \mathcal{D}(\gamma)$. We say that γ is *subset relation* iff $|\mathcal{NS}(\gamma)| = 0$, γ is a *non-subset relation* iff $|\mathcal{S}(\gamma)| = 0$. If γ is a subset or a non-subset relation we say that γ is *homogeneous* and otherwise *heterogeneous.* If $|\gamma| = 1$ then γ is *atomic* and if $|\mathcal{S}(\gamma)| \leq 1$ then γ is

Horn. We extend these definitions to sets of relations in the obvious way. For example, if Γ is a set of DSRs and all $\gamma \in \Gamma$ are Horn, then Γ is Horn.

Let $\gamma = \{X \subseteq Y, X \,\mathsf{disj}\, Z, X \neq W\}$. Then $\mathcal{S}(\gamma) = \{X \subseteq Y\}$, $\mathcal{NE}(\gamma) = \{X \neq W\}$, $\mathcal{D}(\gamma) = \{X \,\mathsf{disj}\, Z\}$ and $\mathcal{NS}(\gamma) = \{X \,\mathsf{disj}\, Z, X \neq W\}$. Furthermore, γ is Horn but not atomic.

The satisfiability problem for DSRs is denoted DSRSat and has the following definition:

Instance: A finite set Γ of atomic DSRs.
Question: Does there exist a model of Γ?

Later on, we will see that DSRSat is NP-complete. However, we begin with the simpler problem AtomDSRSat which is DSRSat restricted to atomic DSRs.

An AtomDSRSat instance can be represented by a labelled directed graph, where the nodes are set variables and the arcs are labelled by the relations $\subseteq$, $\neq$ and disj. From now on, let Γ be an arbitrary instance of AtomDSRSat and $G = \langle V, E\rangle$ the labelled directed graph representing it. Let G' be the graph obtained from G by removing arcs which are not labelled $\subseteq$. To be able to understand the algorithm in Figure 1, we need some auxiliary definitions.

Definition 2.2 A node $Z \in V$ is *null-defined* iff there exist $X, Y \in V$ such that

- there exists an arc $e \in E$ from X to Y labelled disj in G; and
- there exists a path from Z to X in G'; and
- there exists a path from Z to Y in G'.

We allow paths to be empty, *i.e.*, there always exists a path from X to X for arbitrary X. Hence, the case when we have $X \,\mathsf{disj}\, X$ is covered by this definition. The following lemma is immediate.

Lemma 2.3 If G contains a null-defined node, then there exists no S-model of G.

Proof: If Z is null-defined, then it must be interpreted as the empty set. Consequently, there exists no S-model of Z. $\square$

Definition 2.4 Two nodes $X, Y \in V$ are *inconsistent* iff

- there exists an arc $e \in E$ from X to Y labelled $\neq$; and
- there exists a path from X to Y in G'; and
- there exists a path from Y to X in G'.

Lemma 2.5 If G contains two inconsistent nodes, then there exists no S-model of G.

Proof: If X and Y are inconsistent, then every S-model I must assign values to X and Y such that $I(X) = I(Y)$ and $I(X) \neq I(Y)$. Clearly, this implies the non-existence of an S-model of G. $\square$

We can now show the tractability of AtomDSRSat and the NP-completeness of DSRSat.

Theorem 2.6 AtomDSRSat is polynomial.

Proof: The algorithm Atom-SAT is obviously polynomial. We claim that the algorithm accepts iff the given instance has an S-model.

Assume the algorithm rejects in line 3 or line 4. Then there exists no S-model by Lemmata 2.3 and 2.5.

Assume to the contrary that the algorithm accepts. We sketch how to construct a model I of G over the set $\mathbb{N}$ of natural numbers.

Assume $V = \{v_1, \ldots, v_n\}$ and define the function $I : V \to 2^{\mathbb{N}}$ such as

$$I(v_i) = \{i\} \cup \{j \mid \text{ there is a path from } v_j \text{ to } v_i \text{ in } G'\}$$

for $1 \leq i \leq n$. We show that I is a model of G by considering the three ways two variables can be related.

1. $v_i \subseteq v_j$. Then there exists a path from v_i to v_j in G' which implies that $I(v_i) \subseteq I(v_j)$ by the construction of I.
2. $v_i \neq v_j$. First, note that v_i and v_j are not inconsistent since otherwise the algorithm would have rejected in line 3. If i is a member of $I(v_j)$, then $j \notin I(v_i)$ since, otherwise, v_i and v_j would be inconsistent. Hence, $I(v_i) \neq I(v_j)$ in this case. Assume to the contrary that i is not a member of $I(v_j)$. It follows immediately that $I(v_i) \neq I(v_j)$ since $i \in I(v_i)$.

1 **algorithm** Atom-SAT.
2 Input: An instance G of Atom-SAT.
3 **if** there exists a null-defined node in G **then reject**
4 **if** there exist inconsistent nodes in G **then reject**
5 **accept**

1 **algorithm** Horn-SAT.
2 Input: An instance Γ of HORNDSRSAT.
3 $A \leftarrow \bigcup\{\gamma \in \Gamma \mid |\mathcal{NS}(\gamma)| = 0\}$
4 **if** A is not satisfiable **then reject**
5 **if** $\exists \gamma \in \Gamma$ that blocks A and is a non-subset DSR **then reject**
6 **if** $\exists \gamma \in \Gamma$ that blocks A and is heterogeneous **then**
Compute Horn-SAT$((\Gamma - \{\gamma\}) \cup \mathcal{S}(\gamma)$
7 **accept**

Figure 1: The algorithms for ATOMDSRSAT *and* HORNDSRSAT.

3. $v_i \,\mathsf{disj}\, v_j$. Assume there exists a $k \in \mathbb{N}$ such that $k \in I(v_i)$ and $k \in I(v_j)$. By the construction of I, there must exist a path from v_k to v_i in G' and a path from v_k to v_j in G'. Obviously, this implies that v_k is null-defined. This leads to a contradiction since the algorithm did not reject in line 4. Consequently, $I(v_i) \,\mathsf{disj}\, I(v_j)$. □

Theorem 2.7 DSRSAT is NP-complete.

Proof: Membership in NP follows from Theorem 2.6: Given a set Γ of DSRs, pick one set relation from each DSR in Γ and verify (in polynomial time) the satisfiability of these relations.

We prove NP-hardness by a polynomial-time reduction from the NP-complete problem GRAPH 3-COLOURABILITY. Let $G = \langle V, E \rangle$ be an arbitrary undirected graph. We construct a set of DSRs that are S-satisfiable iff G is 3-colourable. First we introduce three sets $C1$, $C2$ and $C3$ which are related as follows: $\{C1 \,\mathsf{disj}\, C2\}$, $\{C1 \,\mathsf{disj}\, C3\}$, $\{C2 \,\mathsf{disj}\, C3\}$. These sets will correspond to the three "colours" to paint the nodes of G with. For each $w \in V$, we introduce a set variable W and the following two DSRs: $\{W \subseteq C1,\ W \subseteq C2,\ W \subseteq C3\}$ and $\{C1 \subseteq W,\ C2 \subseteq W,\ C3 \subseteq W\}$.

This construction forces W to equal exactly one of the sets $C1$, $C2$ or $C3$. Finally, for each arc $\{x, y\} \in E$ we introduce the DSR $\{X \neq Y\}$. It is obvious that the resulting set of DSRs are S-satisfiable iff G is 3-colourable. Hence, DSRSAT is NP-complete. □

3 The Horn Case

We study the S-satisfiability problem for Horn DSRs, denoted HORNDSRSAT, in this section. To show HORNDSRSAT tractable, the next definition is central.

Definition 3.1 Let A be a satisfiable set of atomic DSRs and let γ be a DSR. We say that γ *blocks* A iff for every $d \in \mathcal{NS}(\gamma)$, $A \cup \{d\}$ is not satisfiable.

Clearly, we can use the Atom-SAT algorithm to decide (in polynomial time) whether a DSR γ blocks a set A of atomic DSRs.

We claim that the algorithm in Figure 1 correctly solves HORNDSRSAT in polynomial time. The key lemma in proving the claim is the following.

Lemma 3.2 Let Γ be a set of arbitrary Horn DSRs. Let $A \subseteq \Gamma$ be the set of subset DSRs in Γ and let $D = \{D_1, \ldots, D_k\} \subseteq \Gamma$ be the set of DSRs that are not subset DSRs. Under the condition that A is S-satisfiable, Γ is S-satisfiable if D_i does not block A for any $1 \leq i \leq k$.

Proof: Pick one non-subset relation d_i out of every D_i such that $A \cup \{d_i\}$ is S-satisfiable. This is possible since no D_i blocks A. We show that $\Gamma' = A \cup \{d_1, \ldots, d_k\}$ is S-satisfiable and, hence, Γ is S-satisfiable.

Let G be the equivalent graph representation of Γ'. By the algorithm Atom-SAT, we know that there are only two circumstances that make G not satisfiable: (1) there exists a null-defined node in G; and (2) there exist inconsistent nodes in G. Hence, if G is not S-satisfiable then one of the two cases above apply.

Assume there exists a null-defined node Z in G. By the definition of null-defined, Z is null-defined also in some instance $A \cup \{d_i\}$, $1 \leq i \leq k$. In other words, d_i blocks A which contradicts our initial assumptions. Assume there exist two inconsistent nodes X and Y. Then there exists an instance $\Sigma = A \cup \{d_i\}$, $1 \leq i \leq k$ such that X and Y are inconsistent in Σ. Hence, d_i blocks A which leads to a contradiction. Consequently, Γ' is satisfiable and the lemma follows. □

The soundness and completeness of Horn-SAT are settled in the forthcoming three lemmata.

Lemma 3.3 Let Γ be a set of Horn DSRs and let $C \subseteq \Gamma$ be the set of subset DSRs in Γ. If there exists a heterogeneous DSR $\gamma \in \Gamma$ such that γ blocks C, then Γ is S-satisfiable iff $(\Gamma - \{\gamma\}) \cup \mathcal{S}(\gamma)$ is S-satisfiable.

Proof: *if:* Trivial.
only-if: If Γ is S-satisfiable, then γ has to be S-satisfiable. Since γ blocks C, $\mathcal{S}(\gamma)$ must be S-satisfied in any solution of Γ. $\Box$

Lemma 3.4 Let Γ be a set of Horn DSRs. If Horn-SAT(Γ) accepts then Γ is S-satisfiable.

Proof: Induction over n, the number of heterogeneous DSRs in Γ.
Basis step: If $n = 0$ and Horn-SAT(Γ) accepts then the formulae in A are S-satisfiable and there does not exist any $\gamma \in \Gamma$ that blocks A. Consequently, Γ is S-satisfiable by Lemma 3.2.
Induction hypothesis: Assume the claim holds for $n = k$, $k \geq 0$.
Induction step: Γ contains $k+1$ heterogeneous DSRs. If Horn-SAT accepts in line 6 then $(\Gamma - \{\gamma\}) \cup \mathcal{S}(\gamma)$, which contains k heterogeneous DSRs, is S-satisfiable by the induction hypothesis. By Lemma 3.3, this is equivalent to Γ being S-satisfiable. If Horn-SAT accepts in line 7, then there does not exist any disequational or heterogeneous $\gamma \in \Gamma$ which blocks A. By Lemma 3.2, this means that Γ is S-satisfiable. $\Box$

Lemma 3.5 Let Γ be a set of Horn DSRs. If Horn-SAT(Γ) rejects then Γ is not S-satisfiable.

Proof sketch: Induction over the number of heterogeneous DSRs in Γ. The proof is similar to the proof of Lemma 3.4. $\Box$

Finally, we can show that Horn-SAT is a polynomial-time algorithm and, thus, show that HORNDSRSAT is polynomial.

Theorem 3.6 HORNDSRSAT is polynomial.

Proof: By Lemmata 3.4 and 3.5, it is sufficient to show that Horn-SAT is polynomial. The number of recursive calls is bounded by the number of heterogeneous DSRs in the given input. By Theorem 2.6, we can in polynomial time decide whether a set of atomic DSRs is S-satisfiable. Since we need only check a polynomial number of such systems in each recursion, the theorem follows. $\Box$

4 The Horn Fragment of RCC-5

We follow Bennett [1994] in our definition of RCC-5. RCC-5 is based on the notions *regions* and *binary relations* on them. A region X is a variable interpreted over the non-empty sets. It should be noted that we do not require the sets to be open sets in some topological space. This is no limitation since it is impossible to distinguish interior points from boundary points in RCC-5. Thus we can take any set $\mathcal{U}$ and use the discrete topology $\mathcal{T} = \langle \mathcal{U}, 2^{\mathcal{U}} \rangle$. Then every subset of $\mathcal{U}$ is an open set in $\mathcal{T}$.

We assume that we have a fixed universe of variable names for regions. Then, an *R-interpretation* is a function that maps region variables to the non-empty subsets of some set.

Given two interpreted regions, their relation can be described by exactly one of the elements of the set $\mathbf{B}$ of five *basic RCC-5 relations*. These relations are defined as follows: for all X and Y, $X\{\mathtt{DR}\}Y$ iff $X \cap Y = \emptyset$, $X\{\mathtt{PO}\}Y$ iff $\exists a, b, c : a \in X, a \notin Y, b \in X, b \in Y, c \notin X, c \in Y$, $X\{\mathtt{PP}\}Y$ iff $X \subset Y$, $X\{\mathtt{PP}^{-1}\}Y$ iff $X \supset Y$ and $X\{\mathtt{EQ}\}Y$ iff $X = Y$.

A formula of the form XBY where X and Y are regions and $B \in \mathbf{B}$, is said to be satisfied by an R-interpretation iff the interpretation of the regions satisfies the relations as specified above. To express indefinite information, unions of the basic relations are used, written as sets of basic relations, leading to 2^5 binary RCC-5 relations. Naturally, a set of basic relations is to be interpreted as a disjunction of the basic relations. The set of all RCC-5 relations $2^{\mathbf{B}}$ is denoted by R_5.

A formula of the form $X\{B_1, \ldots, B_n\}Y$ is called a RCC-5 formula. Such a formula is satisfied by an R-interpretation I iff XB_iY is satisfied by I for some i, $1 \leq i \leq n$. A finite set Θ of RCC-5 formulae is said to be *R-satisfiable* iff there exists an R-interpretation I that satisfies every formula of Θ. Such a satisfying R-interpretation is called an *R-model* of Θ. The reasoning problem RSAT has the following definition:

INSTANCE: A finite set Θ of RCC-5 formulae.
QUESTION: Does there exist an R-model of Θ?

We will also consider restricted versions of RSAT where the relations used in formulae in Θ are only from a subset S of R_5. We denote such a problem RSAT(S).

Definition 4.1 Let $\mathbf{B} = \{\mathtt{DR}, \mathtt{PO}, \mathtt{PP}, \mathtt{PP}^{-1}, \mathtt{EQ}\}$. RCC-5 consists of the set

$R_5 = 2^{\mathbf{B}}$ and the operations unary *converse* (denoted by $^{\smile}$), binary *intersection* (denoted by $\cap$) and binary *composition* (denoted by $\circ$). They are defined such that for all X, Y,

$$XR^{\smile}Y \text{ iff } YRX;$$

$$X(R \cap S)Y \text{ iff } XRY \wedge XSY; \text{ and}$$

$$X(R \circ S)Y \text{ iff } \exists Z : (XRZ \wedge ZSY).$$

Next, we introduce a *closure* operation. The closure operation transforms a given subclass of R_5 to one that is polynomially equivalent to the original subclass wrt. satisfiability.

Definition 4.2 Let $\mathcal{S} \subseteq R_5$. Then we denote by $\overline{\mathcal{S}}$ the *closure* of $\mathcal{S}$, defined as the least subalgebra containing $\mathcal{S}$ closed under converse, intersection and composition.

A program for computing closures can be obtained from the authors. The easy proof of the next lemma is omitted.

Lemma 4.3 Let $\mathcal{S} \subseteq R_5$. RSAT($\mathcal{S}$) is polynomial iff RSAT($\overline{\mathcal{S}}$) is polynomial. RSAT($\mathcal{S}$) is NP-complete iff RSAT($\overline{\mathcal{S}}$) is NP-complete.

We will now focus upon the tractable subclass $\widehat{\mathcal{H}}_5$ which contain all relations in R_5 except $\{\mathtt{PP}, \mathtt{PP}^{-1}\}$, $\{\mathtt{DR}, \mathtt{PP}, \mathtt{PP}^{-1}\}$, $\{\mathtt{PP}, \mathtt{PP}^{-1}, \mathtt{EQ}\}$ and $\{\mathtt{DR}, \mathtt{PP}, \mathtt{PP}^{-1}, \mathtt{EQ}\}$. For our purposes, another definition is more convenient. Let

$$\mathcal{H}_B = \{\{\mathtt{PO}\}, \{\mathtt{DR}, \mathtt{EQ}\}, \{\mathtt{PP}, \mathtt{EQ}\}\}.$$

By a machine-assisted analysis, it can easily be shown that $\widehat{\mathcal{H}}_5 = \overline{\mathcal{H}_B}$.

Theorem 4.4 RSAT($\widehat{\mathcal{H}}_5$) is polynomially reducible to HornDSRSat.

Proof: RCC-5 formulae of the form $X\{\mathtt{PP}, \mathtt{EQ}\}Y$ can obviously be replaced by the Horn DSR $X \subseteq Y$. Likewise, an RCC-5 formula of the form $X\{\mathtt{DR}, \mathtt{EQ}\}Y$ can be replaced by the two Horn DSRs $X\{\mathsf{disj}, \subseteq\}Y$ and $Y\{\mathsf{disj}, \subseteq\}X$. Finally, RCC-5 formulae of the type $X\{\mathtt{PO}\}Y$ can be replaced by the following six DSRs: $A\{\subseteq\}X$, $A\{\mathsf{disj}\}Y$, $B\{\subseteq\}X$, $B\{\subseteq\}Y$, $C\{\mathsf{disj}\}X$ and $C\{\subseteq\}Y$, where A, B and C denote fresh set variables.

We have thus shown how $\mathrm{RSAT}(\mathcal{H}_B)$ can be reduced to HORNDSRSAT in polynomial time. Since we know that $\widehat{\mathcal{H}}_5 = \overline{\mathcal{H}_B}$, the theorem follows from Lemma 4.3. $\square$

Note that the Horn DSR framework allows for spatial formulae which are not formulae in RCC-5. For instance, the disjunction $X\{\mathtt{PO}\}Y \vee X\{\mathtt{PP}\}Z$ cannot be expressed in RCC-5, even though both PO and PP are in RCC-5. However, the expression is easily converted to a set of Horn DSRs.

5 Topological Properties of RCC-5

The goal of this section is to study some topological properties of RCC-5. First, we show that RCC-5 can always be interpreted over the open sets of the real line under the natural topology. Then, we study some more restrictive topological assumptions and their impact on the complexity of RCC-5.

The following theorem follows immediately from the model construction in Theorem 2.6.

Theorem 5.1 Let Γ be an S-satisfiable, finite set of DSRs. Then Γ has a model that maps region variables to finite subsets of $\mathbb{N}$.

We say that a model that assigns finite subsets of $\mathbb{N}$ to region variables is a $\mathbb{N}$-model. Recall that the natural topology of $\mathbb{R}$ has the set of all open intervals as its basis. This implies that every finite union of open intervals is an open set under this topology.

Theorem 5.2 Let Θ be a R-satisfiable set of RCC-5 formulae. Then Θ has a model of open sets (under the natural topology) in $\mathbb{R}$.

Proof: Let $\Theta = \{\theta_1, \ldots, \theta_n\}$ be an arbitrary, satisfiable, finite set of RCC-5 formulae. Since Θ is R-satisfiable then for each i, $1 \leq i \leq n$, there exists an atomic RCC-5 relation $\alpha_i \in \gamma_i$ such that $A = \{\alpha_1, \ldots, \alpha_n\}$ is R-satisfiable. Observe that A is an instance of $\mathrm{RSAT}(\widehat{\mathcal{H}}_5)$. Hence, we can polynomially reduce A to an equivalent DSRSAT instance B by Theorem 4.4. By Theorem 5.1, B has a $\mathbb{N}$-model I. Hence, there also exists a model I' from the variables in A to the finite subsets of $\mathbb{N}$. Consider the following interpretation:

$$I''(v) = \{y \in \mathcal{R} \mid x \in I'(v) \wedge x < y < x + 1/2\}.$$

Clearly, I'' is a model of A since I' is a model of A. Furthermore, $I'(v)$ is the union of open intervals in $\mathbb{R}$. Thus, $I'(v)$ is an open set in $\mathcal{R}$ under the natural topology, which concludes the proof. □

It should be clearly understood that the real line equipped with the natural topology is a very weak topological assumption for a spatial theory. Hence, it is interesting to investigate stronger topological assumptions. Such studies have been performed by, for instance, Grigni *et al.* [1995] and Lemon [1996].

Grigni *et al.* [1995] have studied spatial reasoning under such an assumption; they require the regions to be connected sets in two-dimensional Euclidean space. This makes the problem appreciably harder. They show that RCC-5[1] restricted to only atomic relations is NP-hard under these restrictions.

Even more surprising, if we only use the relations DR and PO, the problem remains NP-hard. In this section, we will show that this result does not hold if we allow the regions to be connected sets in three-dimensional Euclidean space (E^3). We say that such a model is a 3DC-model.

Let $R_3 = 2^{\{\mathtt{DR},\mathtt{PO},\mathtt{EQ}\}}$. We define the computational problem, which we will denote R3SAT, as follows:

INSTANCE: A finite set Θ of R_3 formulae.
QUESTION: Does there exist an 3DC-model of Θ?

Our initial observation is that if a set Θ of R_3 formulae has a 3DC-model, then it has an R-model (trivially). By showing the converse, *i.e.* if Θ has an R-model then it has a 3DC-model, we have also shown the polynomiality of R3SAT since $R_3 \subseteq \widehat{\mathcal{H}}_5$.

In the next lemma, we show how to modify an $\mathbb{N}$-model into a 3DC-model.

Lemma 5.3 Let Θ be a finite set of R_3 formulae with an $\mathbb{N}$-model. Then there exists a 3DC-model of Θ.

Proof: Let S be a finite set of natural numbers. We begin by constructing a three-dimensional connected set fork(S) with certain properties. The definition of fork is as follows:

$$\begin{aligned}\mathrm{fork}(S) \quad = \quad & \{(s,y,0) \mid s \in S \wedge 0 \leq y \leq 1\} \cup \\ & \{(x,1,0) \mid \min(S) \leq x \leq \max(S)\}.\end{aligned}$$

[1] In their terminology, RCC-5 is named "medium resolution".

An example of a fork set can be found in Figure 2. Clearly, fork sets are connected.

Let I be an $\mathbb{N}$-model of Θ and assume Θ contains the region variables $X = \{X_1, \ldots, X_n\}$. Let $\alpha : X \rightarrow \{y \in \mathcal{R} \mid 0 \leq y < 2\pi\}$ be an arbitrary function satisfying the following requirement: $\alpha(X_i) = \alpha(X_j)$ iff $I(X_i) = I(X_j)$. We shall now construct a 3DC-model I' of Θ: define $I'(X_i)$ as $\text{fork}(I(X_i))$ rotated $\alpha(X_i)$ radians around the x-axis.

It remains to show that I' is a model of Θ. We have three cases:

1. $X_i\{\texttt{DR}\}X_j$ under I. Then $I(X_i) \neq I(X_j)$ so $\alpha(X_i) \neq \alpha(X_j)$. Consequently, $X_i\{\texttt{DR}\}X_j$ under I'.

2. $X_i\{\texttt{PO}\}X_j$ under I. Then there exist natural numbers a, b, c such that

$$a \in I(X_i), a \notin I(X_j)$$
$$b \in I(X_i), b \in I(X_j)$$
$$c \notin I(X_i), c \in I(X_j)$$

By the construction of fork, it follows that

$$(a, 0, 0) \in \text{fork}(I(X_i)), (a, 0, 0) \notin \text{fork}(I(X_j))$$
$$(b, 0, 0) \in \text{fork}(I(X_i)), (b, 0, 0) \in \text{fork}(I(X_j))$$
$$(c, 0, 0) \notin \text{fork}(I(X_i)), (c, 0, 0) \in \text{fork}(I(X_j)).$$

Now, observe that the points we refer to are on the x-axis which is invariant under the rotation imposed by α. Hence, $X_i\{\texttt{PO}\}X_j$ under I'.

3. $X_i\{\texttt{EQ}\}X_j$ under I. Then $I(X_i) = I(X_j)$ so $\alpha(X_i) = \alpha(X_j)$. Furthermore, $\text{fork}(I(X_i)) = \text{fork}(I(X_j))$ which implies that $X_i\{\texttt{DR}\}X_j$ under I'.

We have thus shown that I' is a 3DC-model of Θ and the lemma follows. □

Theorem 5.4 R3SAT is polynomial.

Proof: Let Θ be an instance of R3SAT. Since $R_3 \subseteq \widehat{\mathcal{H}}_5$, it is tractable to decide whether Θ has an R-model or not. If Θ does not have an R-model, then Θ cannot have a 3DC-model. Assume to the contrary that Θ has an R-model. Then, by Theorem 5.1, Θ has an $\mathbb{N}$-model. This implies, by Lemma 5.3, the existence of a 3DC-model of Θ. □

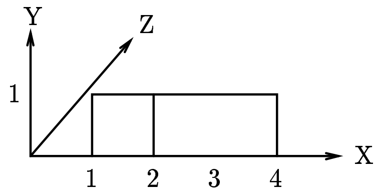

Figure 2: The set fork$(\{1, 2, 4\})$.

6 Discussion

Results by Drakengren and Jonsson [1996] show that algorithms similar to Horn-SAT can be used for tractable inference in intuitionistic logic. This opens the way for some intriguing new research. Bennett [1994] has shown that a subtheory of RCC, RCC-8, can be reduced to propositional intuitionistic logic. RCC-8 is more expressive than RCC-5 since it makes it possible to distinguish between boundary and interior points. There are, for instance, two versions of the `PP` relation, one that allow boundary contact and one that disallows such contact.

By employing the connection between RCC-8 and intuitionistic logic established by Bennett, Nebel [1995] discovered the first published tractable subclass of RCC-8. These results were extended by Renz [1996] who isolated a large tractable subclass of RCC-8, denoted $\widehat{\mathcal{H}}_8$. This was accomplished by studying the connection between RCC-8 and yet another logic, a *multimodal* logic. Probably, there is a connection between spatial reasoning, qualitative reasoning about sets and (intuitionistic and/or modal) logic which certainly deserves further study.

7 Conclusions

We have proposed a tractable algorithm for qualitative reasoning about sets. The algorithm can decide the satisfiability of set variables related with subset, disjointness and non-equality. Furthermore, the algorithm allows disjunctions of such expressions under one restriction: there can be at most one subset relation in each disjunction.

To demonstrate the algorithm, we have applied it to problems in spatial reasoning. We have thus shown the existence of a straightforward, polynomial-time algorithm for the Horn fragment of RCC-5 [Renz, 1996]. Furthermore, we have found a subclass of RCC-5, R_3, for which it is tractable to decide whether there exist connected sets in three-dimensional Euclidean space satisfying the given restrictions or not. This finding complements the results in Grigni *et al.* [1995] showing a subclass of R_3 to be NP-complete if we require the sets to be connected sets in two-dimensional Euclidean space.

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Paper VIII

Reasoning About Set Constraints Applied to Tractable Inference in Intuitionistic Logic

Thomas Drakengren and Peter Jonsson

ABSTRACT

Automated reasoning about sets has received a considerable amount of interest in the literature. Techniques for such reasoning have been used in, for instance, analyses of programming languages, terminological logics and spatial reasoning. In this paper, we identify a new class of set constraints where checking satisfiability is tractable (i.e. polynomial-time). We show how to use this tractability result for constructing a new tractable fragment of intuitionistic logic. Furthermore, we prove NP-completeness of several other cases of reasoning about sets.

1 Introduction

There has been considerable interest in formalisms for describing and reasoning about sets. We begin by describing some of these. The most well-studied class of set constraints is, probably, *Herbrand set constraints.* Such have been suggested as a formalism for describing relationships between sets of terms of a free algebra. A *positive* set constraint has the form $X \subseteq Y$, where X and Y are *set expressions.* Examples of set expressions are $\varnothing$ (the empty set), α (a set-valued variable), $c(X, Y)$ (a constructor application), and the union, intersection, or complement of set expressions. The computational problem then is to decide whether there exists an assignment that satisfies all the given set constraints. This is the classical definition of a set constraint as stated in, for example, the paper by Heintze and Jaffar [1990]. However, certain researchers have also studied *negative* set constraints [Marriott and Odersky, 1992; Aiken and Murphy, 1991], i.e. set constraints of the form $X \not\subseteq Y$.

There are a number of important applications for automated reasoning about Herbrand set constraints. These include analyses of programming languages [Aiken *et al.*, 1994b; Heintze and Jaffar, 1990; Reynolds, 1969], logic programming [Aiken and Lakshman, 1994; Heintze and Jaffar, 1992] and the study of finite automata [Brzozowski and Leiss, 1980]. On the theoretical side, rapid progress has been made in understanding the algorithms for and complexity of solving various classes of set constraints. One of the most important results is that the satisfiability problem for set constraint is decidable even if we allow negative constraints [Aiken *et al.*, 1994a; Gilleron *et al.*, 1993]. On the practical side, several systems have been implemented for reasoning based on solving systems of set constraints [Aiken *et al.*, 1994b; Heintze, 1992].

A variant of set constraints are the *Tarskian set constraints* [Givan *et al.*, 1996; Jónsson and Tarski, 1951; Jónsson and Tarski, 1952]. Syntactically, Tarskian set constraints are very similar to Herbrand set constraints. The difference is that Tarskian set constraints are interpreted relative to a first-order structure whereas Herbrand set constraints are always interpreted over the Herbrand universe generated by a specified set of constructor functions. This difference fundamentally changes the nature of the satisfiability problem.

Reasoning about sets has also received attention in the artificial intelligence community. One example is so-called "concept languages" or "terminological languages" [Schmidt-Schauß and Smolka, 1991; Brachman and

Schmolze, 1985]. The set expressions of concept languages are constructed from from set variables and relation variables using a variety of compositional mechanisms. For instance, the expression $\forall R.C$ where R is a relation expression and C is a set expression denotes the set

$$\{x \mid \forall y.R(x,y) \rightarrow y \in C\}.$$

Many of these languages can be viewed as highly restricted fragments of propositional dynamic logic (PDL) [Donini *et al.*, 1991]. Thus, many of them have satisfiability problems in P, NP or PSPACE which is significantly better than full PDL which is EXPTIME-complete [De Giacomo and Lenzerini, 1994].

Reasoning about sets has also appeared in connection with other types of logics, such as modal logics. These logics involve formulae which are true or false of possible worlds in Kripke structures. It is natural to view a Kripke structure as an ordinary first order structure whose domain is the set of possible worlds. In this view each formula denotes a set of worlds, which is nothing more than a subset of the domain of the structure in which the formula is interpreted.

In this paper, we study a variation of reasoning about Tarskian set constraints. More specifically, we continue the study of the set relations by Jonsson and Drakengren [1996], restricting the set expressions to only allow for set-valued variables, but allowing them to be related by relations other than $\subseteq$: variables are related with subset ($\subseteq$), disjointness (disj) and non-equality ($\neq$), and we allow disjunctions of such set relations. As a concrete example, consider the following three relations:

$$Z \subseteq X, Z \subseteq Y, X \,\mathsf{disj}\, Y.$$

It is not hard to see that we can satisfy these relations by setting $X = Y = Z = \varnothing$. However, if we disallow the use of the empty set, we cannot satisfy the relations. The case investigated by Jonsson and Drakengren [1996] was the case where the empty set is disallowed, whereas the emphasis of this paper is on the other case, allowing the empty set as well. We shall see that this choice has a strong impact on the complexity of the satisfiability problem in certain cases.

The results in this paper complement the results of Jonsson and Drakengren [1996], and show the complexity of the satisfiability problem for five types of formulae, where the empty set is allowed in models of set contraints: general formulae, Horn formulae of two kinds, Krom formulae and atomic formulae. The results show that the problem is polynomial for atomic formulae and NP-complete for Krom and general formulae. For one kind of Horn formulae, the problem is polynomial if we disallow the empty set and NP-complete otherwise, but for the other kind both problems are polynomial.

Even though the problem that we are about to study appears to be severely restricted, it appears to be useful. To demonstrate this, we show how it can be used in *inference in intuitionistic logic.* We exhibit a subclass of intuitionistic propositional logic (see, for example, Heyting [1971] or Dragalin [1988] for an introduction), where computing entailment can be done in polynomial time. This an improvement of the general case, since the full problem has been shown to be PSPACE-complete [Statman, 1979]. Furthermore, intuitionistic logic and spatial reasoning are intimately connected: it is used for that purpose by Bennett [1994], and indeed, Nebel [1995] found a tractable subclass of intuitionistic logic, which aided in finding a tractable subclass of spatial reasoning. In this paper, we show that our tractable class is incomparable to that of Nebel [1995].

The structure of the paper follows: In Section 2 we introduce the formalism to be used, the *disjunctive set relations*, DSRs. Its computational properties are investigated for different restrictions on the DSRs in Section 3, Section 4 and Section 5. Finally, we apply these results to intuitionistic logic in Section 6, after which a discussion and conclusions conclude the paper.

2 Disjunctive Set Relations

Our approach to reasoning about sets is based on that of Jonsson and Drakengren [1996], extended with concepts for handling also empty sets in models: There are *set variables* and *relations* on these. We assume that we have a fixed universe of variable names for sets. Then, an $S_\varnothing$*-interpretation* is a function that maps set variables to (possibly empty) sets whereas an S*-interpretation* is a function that maps set variables to nonempty sets. The case of S-interpretations has already been investigated by Jonsson and Drakengren [1996].

We will relate set variables by the following three relations: subset ($\subseteq$), non-equality ($\neq$) and disjointness (disj) having the usual definitions. A formula of this type is called an *atomic relation.* A *disjunctive set relation* (DSR) is a set (disjunction) of atomic relations.

An atomic relation XRY is satisfied by an $S_\varnothing$-interpretation (S-interpretation) I iff $I(X)RI(Y)$. A DSR $\gamma = \{\gamma_1, \ldots, \gamma_n\}$ (which is more conveniently written as a disjunction $\gamma = \gamma_1 \vee \ldots \vee \gamma_n$) is said to be $S_\varnothing$-satisfiable (S-satisfiable) iff there exists an $S_\varnothing$-interpretation (S-interpretation) that satisfies at least one member of each γ_i, $1 \leq i \leq n$. Such a satisfying $S_\varnothing$-interpretation (S-interpretation) is said to be an $S_\varnothing$-model (S-model) of γ. We also allow conjunctions of two or more DSRs $\alpha_1, \ldots, \alpha_n$ written as $\alpha_1 \wedge \ldots \wedge \alpha_n$, which are interpreted in the obvious way.

For DSRs, we have the following natural reasoning problem which we will denote DSRSAT$_\varnothing$ and DSRSAT, respectively.

Relation	DSR-form
$X = \varnothing$	$X \,\mathsf{disj}\, X$
$X \neq \varnothing$	$E \,\mathsf{disj}\, E \wedge X \neq E$
$X = Y$	$X \subseteq Y \wedge Y \subseteq X$
$X \subset Y$	$X \subseteq Y \wedge X \neq Y$
$X \not\subseteq Y$	$F \neq \varnothing \wedge F \subseteq X \wedge F \,\mathsf{disj}\, Y$
$X \cap Y \neq \varnothing$	$F \neq \varnothing \wedge F \subseteq X \wedge F \subseteq Y$
$X \subseteq Y \cap Z$	$X \subseteq Y \wedge X \subseteq Z$
$Y \cup Z \subseteq X$	$Y \subseteq X \wedge Z \subseteq X$

Table 1: Some relations that can be expressed as DSRs. E and F denote fresh set variables.

INSTANCE: A finite set Γ of DSRs.
QUESTION: Does there exist an $S_\varnothing$-model (S-model) of Γ?

We will see that DSRSAT$_\varnothing$ is NP-complete later on. DSRSAT was proved to be NP-complete by Jonsson and Drakengren [1996]. To define tractable subclasses of these problems, we need to name some special classes of DSRs.

Definition 2.1 Let γ be a DSR. Then we let $\mathcal{S}(\gamma)$ denote the set of subset relations in γ, $\mathcal{NE}(\gamma)$ the set of non-equality relations, $\mathcal{D}(\gamma)$ the set of disjointness relations, $\mathcal{NS}(\gamma) = \mathcal{NE}(\gamma) \cup \mathcal{D}(\gamma)$, and $\mathcal{SD}(\gamma) = \mathcal{S}(\gamma) \cup \mathcal{D}(\gamma)$. We say that γ is *subset relation* iff $|\mathcal{NS}(\gamma)| = 0$, γ is a *non-subset relation* iff $|\mathcal{S}(\gamma)| = 0$, γ is an *$\mathcal{SD}$-relation* iff $|\mathcal{NE}(\gamma)| = 0$, and γ is a *non-$\mathcal{SD}$-relation* iff $|\mathcal{SD}(\gamma)| = 0$. If γ is a subset or a non-subset relation we say that γ is *homogeneous* and otherwise *heterogeneous*. If γ is an $\mathcal{SD}$ or a non-$\mathcal{SD}$-relation we say that γ is *2-homogeneous* and otherwise *2-heterogeneous*. If $|\gamma| = 1$, then γ is *atomic*, if $|\gamma| \leq 2$, then γ is *Krom*, if $|\mathcal{S}(\gamma)| \leq 1$, then γ is *Horn*, and if $|\mathcal{SD}(\gamma)| \leq 1$, then γ is *2-Horn*. It is easy to see that if γ is 2-Horn, then γ is also Horn. We extend these definitions to sets of relations in the obvious way. For example, if Γ is a set of DSRs and all $\gamma \in \Gamma$ are 2-Horn, then Γ is 2-Horn. □

Some examples of DSRs follow.

Example 2.2 Let

$$\gamma \equiv X \neq Y \vee Y \,\mathsf{disj}\, Z \vee W \subseteq X \vee X \subseteq Y.$$

Now

$$\mathcal{S}(\gamma) = W \subseteq X \vee X \subseteq Y,$$

$$\mathcal{NE}(\gamma) = X \neq Y,$$

$$\mathcal{D}(\gamma) = Y \,\mathsf{disj}\, Z,$$

$$\mathcal{NS}(\gamma) = X \neq Y \vee Y \,\mathsf{disj}\, Z,$$

and

$$\mathcal{SD}(\gamma) = W \subseteq X \vee X \subseteq Y \vee Y \,\mathsf{disj}\, Z.$$

We see that γ is not Horn, and also not 2-Horn or Krom. The formula

$$X \subseteq Y \vee Y \subseteq Z$$

is a Krom DSR,

$$X \subseteq Y \vee W \neq Z \vee W \,\mathsf{disj}\, X$$

is a Horn DSR, but not a 2-Horn DSR, and

$$X \subseteq Y \vee W \neq Z \vee Z \neq V$$

and

$$X \,\mathsf{disj}\, Y \vee X \neq Z \vee Y \neq W$$

are 2-Horn DSRs. □

From the relations $\subseteq$, disj and $\neq$, several other set relations can be constructed. Table 1 contains a few examples.

3 The Atomic Case

The first reasoning problem that we will study is ATOMDSRSAT$_\varnothing$:

INSTANCE: A finite set Γ of atomic DSRs.
QUESTION: Does there exist an $S_\varnothing$-model of Γ?

We claim that Algorithm 3.1 correctly solves ATOMDSRSAT$_\varnothing$, and furthermore, does so in polynomial time. The corresponding algorithm for S-interpretations was proved correct and polynomial by Jonsson and Drakengren [1996].

Algorithm 3.1 (Atom-SAT$_\varnothing(G)$)

input instance $G = \langle V, E \rangle$ of Atom-SAT$_\varnothing$

Extend G with a new node $\varnothing$ and a new edge $\varnothing \,\mathsf{disj}\, \varnothing$
while there exists a null-defined node $Z \in V$ different from $\varnothing$
    collapse the nodes Z and $\varnothing$ in G
if there exist inconsistent nodes $X, Y \in V$ **then**
    reject
accept

$\square$

To understand the algorithm Atom-SAT$_\varnothing$, a few things need to be explained. First, note that the input to the algorithm does not consist of a finite set of atomic DSRs, but of a directed, labelled graph (DLG). This is no loss of generality since any ATOMDSRSAT$_\varnothing$ instance can be represented by a DLG, where the nodes are set variables and the arcs are labelled by the relations $\subseteq$, $\neq$ and disj. From now on, let Γ be an arbitrary instance of ATOMDSRSAT$_\varnothing$ and $G = \langle V, E \rangle$ the DLG representing it.

We need the following auxiliary definitions.

Definition 3.2 Let $H = \langle W, F \rangle$ be a DLG, $x, y \in W$, and H' the graph obtained from H by removing all edges not labelled by $\subseteq$. Now, a directed path from x to y in H' is said to be a $\subseteq$-*path* in H. $\square$

Next, we defined the concepts of *null-defined* and *inconsistent* nodes.

Definition 3.3 A node $Z \in V$ is *null-defined* iff there exist $X, Y \in V$ such that

- there exists an arc $e \in E$ from X to Y labelled disj in G
- there exists a $\subseteq$-path from Z to X in G'
- there exists a $\subseteq$-path from Z to Y in G'.

□

Note that the case when we have $X \,\mathsf{disj}\, X$ is covered by this definition (we allow zero-length paths).

Definition 3.4 Two nodes $X, Y \in V$ are *inconsistent* iff

- there exists an arc $e \in E$ from X to Y labelled $\neq$
- there exists a $\subseteq$-path from X to Y in G
- there exists a $\subseteq$-path from Y to X in G.

□

The following lemma is immediate and left without proof.

Lemma 3.5 If Z is an null-defined node in G, then Z must be assigned $\varnothing$ by any $S_\varnothing$-model of G, and if X and Y are inconsistent in G, then G has no $S_\varnothing$-model. □

We can now show the completeness of Algorithm 3.1.

Lemma 3.6 Let Γ be a set of atomic DSRs. If Atom-SAT$_\varnothing(\Gamma)$ rejects, then Γ is not $S_\varnothing$-satisfiable.
Proof: First note that the collapsing done in lines 2-3 preserves soundness, i.e. it does not remove any models, by Lemma 3.5.

Suppose that the algorithm rejects, and let G_* be obtained from G by performing the collapsing done in lines 2-3. Then there are two inconsistent nodes in G_*. Now, by Lemma 3.5, G_* cannot have an $S_\varnothing$-model. Since lines 2-3 preserve soundness, G cannot have an $S_\varnothing$-model. □

Next, we prove the soundness of the algorithm.

Lemma 3.7 Let Γ be a set of atomic DSRs. If Atom-SAT$_\varnothing(\Gamma)$ accepts, then Γ is $S_\varnothing$-satisfiable.

Proof: Suppose that the algorithm accepts. Let G_* be the graph obtained from G by performing lines 2-3 of Algorithm 3.1. Note that if G_* has an $S_\varnothing$-model, then G will have an $S_\varnothing$-model, by assigning the empty set to all nodes collapsed with the node $\varnothing$.

We show how to construct an $S_\varnothing$-model I of G_* over the set $\mathbb{N}$ of natural numbers.

Let G' be the graph obtained from G_* by removing arcs which are not labelled $\subseteq$. Assume $V = \{v_\varnothing, v_1, \ldots, v_n\}$ and define the function $I : V \to 2^{\mathbb{N}}$ such that

$$I(v_\varnothing) = \varnothing$$

and for $i \neq \varnothing$

$$I(v_i) = \{i\} \cup \{j \in \mathbb{N} \mid \text{there exists a path from } v_j \text{ to } v_i \text{ in } G'\}$$

for $1 \leq i \leq n$. We will show that I is a model of G_* by considering the three possible ways of relating two variables v_i and v_j in G_*, for $i, j \in \mathbb{N} \cup \{\varnothing\}$.

1. $v_i \subseteq v_j$. Then there exists a path from v_i to v_j in G' which implies $I(v_i) \subseteq I(v_j)$ by the construction of I.

2. $v_i \neq v_j$. First note that $i \neq j$, since otherwise the algorithm would have rejected.

   Suppose that one of v_i or v_j is $v_\varnothing$. Since $i \neq j$, and the only variable which is assigned $\varnothing$ in I is $v_\varnothing$, $I(v_i) \neq I(v_j)$ holds.

   Now suppose that $i, j \in \mathbb{N}$, and assume $i \in I(v_j)$. Then there is a path in G' from v_i to v_j, by the definition of I. But since v_i and v_j are not inconsistent, we must have $j \notin I(v_i)$. Similarly, if $j \in I(v_i)$, then $i \notin I(v_j)$. Thus $I(v_i) \neq I(v_j)$ holds.

3. $v_i \,\mathsf{disj}\, v_j$. If either of v_i or v_j is $v_\varnothing$, the relation holds trivially in I. Thus suppose $i, j \in \mathbb{N}$, and let $k \in I(v_i) \cap I(v_j)$. By the construction of I, there exists a path from v_k to v_i and from v_k to v_j in G'. Thus v_k is null-defined. But since all null-defined nodes have been collapsed with $\varnothing$ in lines 2-3, we must have $v_k = v_\varnothing$, contradicting our assumption. Thus there is no such k, and thus $I(v_i) \,\mathsf{disj}\, I(v_j)$.

□

Theorem 3.8 ATOMDSRSAT$_\varnothing$ is polynomial.
Proof: Correctness of algorithm Atom-SAT follows from Lemma 3.6 and Lemma 3.7. The number of iterations in lines 2-3 is bounded from above by the number of nodes in the given instance, and the tests can easily be carried out in polynomial time. Thus, the theorem follows. □

We conclude this section by showing that DSRSAT$_\varnothing$ and DSRSAT are members of NP, which will be needed later on.

Lemma 3.9 DSRSAT$_\varnothing$ and DSRSAT are in NP.
Proof: Let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ be a finite set of DSRs. If Γ is $S_\varnothing$-satisfiable, then for each i, $1 \leq i \leq n$, there exists an atomic set relation $\alpha_i \in \gamma_i$ such that $A = \{\alpha_1, \ldots, \alpha_n\}$ is $S_\varnothing$-satisfiable. Checking the $S_\varnothing$-satisfiability of A is polynomial by Theorem 3.8. Thus, given Γ we can let A be the non-deterministic guess which can be verified in polynomial time. As a consequence, DSRSAT$_\varnothing$ is in NP.

The proof for DSRSAT is analogous. □

4 The Horn Case

The reasoning problems we shall study in this section extends the class of atomic DSRs, as was done for S-interpretations in Jonsson and Drakengren [Jonsson and Drakengren, 1996]. They are defined as follows:

HORNDSRSAT$_\varnothing$:
INSTANCE: A finite set Γ of Horn DSRs.
QUESTION: Does there exist an $S_\varnothing$-model of Γ?

HORNDSRSAT:
INSTANCE: A finite set Γ of Horn DSRs.
QUESTION: Does there exist an S-model of Γ?

2HORNDSRSAT:
INSTANCE: A finite set Γ of 2-Horn DSRs.
QUESTION: Does there exist an $S_\varnothing$-model of Γ?

2HORNDSRSAT$_\varnothing$:
INSTANCE: A finite set Γ of 2-Horn DSRs.
QUESTION: Does there exist an S-model of Γ?

The problem HornDSRSat was proved to be polynomial by Jonsson and Drakengren [1996], and thus 2HornDSRSat is also polynomial.

Somewhat surprisingly, including the empty set makes the corresponding problem harder: We now show that $\text{HornDSRSat}_{\varnothing}$ is NP-complete.

Theorem 4.1 $\text{HornDSRSat}_{\varnothing}$ is NP-complete, even if we do not allow $\subseteq$ to appear in the formulae.

Proof: Membership in NP follows from Lemma 3.9. Hardness for NP is shown by a polynomial-time reduction from 3sat [Garey and Johnson, 1979]. The problem has the following definition:

Instance: Set U of variables, collection C of clauses over U such that each clause $c \in C$ has $|c| = 3$.
Question: Is C satisfiable?

Let $L = \langle U, C \rangle$ be an instance of 3sat with $U = \{u_1, \ldots, u_m\}$ and $C = \{c_1, \ldots, c_n\}$ We will construct an instance of $\text{HornDSRSat}_{\varnothing}$ that is $S_{\varnothing}$-satisfiable iff L is satisfiable.

We begin by introducing the empty set by the following axiom: $E \,\mathsf{disj}\, E$. Clearly, $E = \varnothing$ in any model. For each u_i in U, we introduce a set variable x_i. We will interpret x_i as follows: u_i is false if $x_i = \varnothing$ and u_i is true if $x_i \neq \varnothing$. Consider the following set of Horn DSRs:

$$\Gamma = \bigcup_{c \in C} \{ \bigvee_{u_p \in c} x_p \neq E \vee \bigvee_{\neg u_p \in c} x_p \,\mathsf{disj}\, x_p \}.$$

Clearly, Γ has an $S_{\varnothing}$-model iff L is satisfiable. Also note that the relation $\subseteq$ is not used anywhere in Γ. Hence, the theorem follows. $\Box$

As a simple consequence of this result and Lemma 3.9, $\text{DSRSat}_{\varnothing}$ is also NP-complete.

Next, we show the tractability of 2HornDSRSat. The proof is very similar to how the tractability of HornDSRSat was proved [Jonsson and Drakengren, 1996]. The idea of constructing Horn classes in this way originated in the paper by Jonsson and Bäckström [1996].

Definition 4.2 Let A be an $S_{\varnothing}$-satisfiable set of atomic DSRs and let γ be a DSR. We say that γ *2-blocks* A iff for every $d \in \mathcal{NE}(\gamma)$, $A \cup \{d\}$ is not $S_{\varnothing}$-satisfiable. $\Box$

Clearly, we can use the AtomDSRSat$_\varnothing$ algorithm to decide in polynomial time whether a DSR γ 2-blocks a set A of atomic DSRs. The next lemma relates satisfiability of a set of 2-Horn DSRs to the satisfiability of a set of atomic DSRs.

We claim that Algorithm 4.3 correctly solves 2HornDSRSat in polynomial time. The next three lemmata prove that the algorithm is sound.

Algorithm 4.3 (2-Horn-SAT(Γ))

input instance Γ of 2HornDSRSat

$A \leftarrow \bigcup\{\gamma \in \Gamma \mid |\mathcal{NE}(\gamma)| = 0\}$
if not Atom-SAT$_\varnothing(A)$ **then**
  reject
if $\exists\gamma \in \Gamma$ that 2-blocks A and is a non-$\mathcal{SD}$ DSR **then**
  reject
if $\exists\gamma \in \Gamma$ that 2-blocks A and is 2-heterogeneous **then**
  2-Horn-SAT$((\Gamma - \{\gamma\}) \cup \mathcal{SD}(\gamma)$
accept

□

Lemma 4.4 Let Γ be a set of 2-Horn DSRs. Let $A \subseteq \Gamma$ be the set of $\mathcal{SD}$ DSRs (which are then atomic, since they are also 2-Horn) in Γ and let $D = \{D_1, \ldots, D_k\} \subseteq \Gamma$ be the set of DSRs that are not $\mathcal{SD}$ DSRs. If A is $S_\varnothing$-satisfiable and D_i does not 2-block A for any $1 \le i \le k$, then Γ is $S_\varnothing$-satisfiable.

Proof: Pick one non-$\mathcal{SD}$ relation d_i out of every D_i such that $A \cup \{d_i\}$ is $S_\varnothing$-satisfiable. This is possible since no D_i 2-blocks A. We shall show that $\Gamma' = A \cup \{d_1, \ldots, d_k\}$ (which we note is a set of atomic relations) is $S_\varnothing$-satisfiable and, hence, Γ is $S_\varnothing$-satisfiable.

Suppose to the contrary that Γ' is not $S_\varnothing$-satisfiable, and let G be the equivalent graph representation of Γ'. Denote by G_* the graph obtained from G by performing lines 1-3 of Algorithm 3.1. By the correctness of Algorithm 3.1, we know that there exist nodes X, Y in G_* which are inconsistent, and thus are connected by an $\neq$ arc in G_*. Since lines 1-3 in the algorithm do not add any $\neq$-relations, there have to exist nodes X', Y' in G connected by $\neq$, which after the collapsing correspond to nodes X and Y in

G_*. This relation has to be a d_i for some i. Consider the graph G' obtained from G by removing all $\neq$-relations except for this d_i. Now run Algorithm 3.1 on G'. Since the collapsings performed in lines 1-3 do not depend on any of the relations removed from G, X and Y will still be inconsistent in the resulting graph G'_*, and thus, $A \cup \{d_i\}$ is unsatisfiable, contradicting the assumption. Thus Γ' is $S_\varnothing$-satisfiable, and hence the same holds for Γ. $\square$

Lemma 4.5 Let Γ be a set of 2-Horn DSRs and let $C \subseteq \Gamma$ be the set of $\mathcal{SD}$ DSRs in Γ. If there exists a 2-heterogeneous DSR $\gamma \in \Gamma$ such that γ 2-blocks C, then Γ is $S_\varnothing$-satisfiable iff $(\Gamma - \{\gamma\}) \cup \mathcal{SD}(\gamma)$ is $S_\varnothing$-satisfiable.
Proof:
$\Leftarrow$) Trivial.
$\Rightarrow$) If Γ is $S_\varnothing$-satisfiable, then γ has to be $S_\varnothing$-satisfiable. Since γ 2-blocks C, $\mathcal{SD}(\gamma)$ must be $S_\varnothing$-satisfied in any solution of Γ. $\square$

Lemma 4.6 Let Γ be a set of 2-Horn DSRs. If 2-Horn-SAT(Γ) accepts, then Γ is $S_\varnothing$-satisfiable.
Proof: Induction over n, the number of 2-heterogeneous DSRs in Γ.

Suppose $n = 0$, and that the algorithm accepts, which has to be in line 8. Then the formulae in A are $S_\varnothing$-satisfiable, and there does not exist any $\gamma \in \Gamma$ that 2-blocks A. Now, by Lemma 4.4, Γ is $S_\varnothing$-satisfiable.
Suppose Γ contains $n+1$ 2-heterogeneous DSRs, assuming that the result holds for n. First suppose that the algorithm accepts in line 7 (that is, within that call). Then $(\Gamma - \{\gamma\}) \cup \mathcal{SD}(\gamma)$, containing n 2-heterogeneous DSRs, is $S_\varnothing$-satisfiable by the induction hypothesis. By Lemma 4.5, this is equivalent to Γ being $S_\varnothing$-satisfiable. Now instead suppose that the algorithm accepts in line 8. Then there does not exist any $\gamma \in \Gamma$ with $|\mathcal{NE}(\gamma)| > 0$ which 2-blocks A. By Lemma 4.4, this means that Γ is $S_\varnothing$-satisfiable. $\square$

The completeness of 2-Horn-SAT is shown in the following two lemmata.

Lemma 4.7 Let Γ be a set of 2-Horn DSRs. Let $C \subseteq \Gamma$ be the set of $\mathcal{SD}$ DSRs in Γ. If there exists a non-$\mathcal{SD}$ DSR $\gamma \in \Gamma$ that 2-blocks C, then Γ is not $S_\varnothing$-satisfiable.
Proof: In any solution to Γ, the relations in $C \cup \{\gamma\}$ have to be be $S_\varnothing$-satisfied. Since γ is a non-$\mathcal{SD}$ DSR and 2-blocks C, this is not possible, and the lemma follows. $\square$

Lemma 4.8 Let Γ be a set of 2-Horn DSRs. If 2-Horn-SAT(Γ) rejects, then Γ is not $S_\varnothing$-satisfiable.

Proof: Induction over n, the number of 2-heterogeneous DSRs in Γ.
If $n = 0$ then, 2-Horn-SAT can reject in lines 3 or 5. If 2-Horn-SAT rejects in line 3, then, trivially, Γ is not $S_\varnothing$-satisfiable. If 2-Horn-SAT rejects in line 5, then there exists a non-$\mathcal{SD}$ DSR $\gamma \in \Gamma$ that 2-blocks A. Hence, Γ is not $S_\varnothing$-satisfiable by Lemma 4.7.
Suppose that Γ contains $n+1$ 2-heterogeneous DSRs, assuming that the result holds for n. If 2-Horn-SAT rejects in line 3, then clearly Γ is not $S_\varnothing$-satisfiable. If 2-Horn-SAT rejects in line 5, then Γ is not $S_\varnothing$-satisfiable by Lemma 4.7. If 2-Horn-SAT rejects in line 7, then $(\Gamma - \{\gamma\}) \cup \mathcal{SD}(\gamma)$, which contains n 2-heterogeneous DSRs, is not $S_\varnothing$-satisfiable by the induction hypothesis. By Lemma 4.5, this is equivalent with Γ not being $S_\varnothing$-satisfiable. □

Finally, we can show that 2-Horn-SAT is a polynomial-time algorithm and, thus, show that 2HornDSRSat is polynomial.

Theorem 4.9 2HornDSRSat is polynomial.
Proof: By Lemma 4.6 and Lemma 4.8, it is sufficient to show that 2-Horn-SAT is polynomial. The number of recursive calls is bounded by the number of 2-heterogeneous DSRs in the given input. By Theorem 3.8, we can in polynomial time decide whether a set of atomic DSRs is $S_\varnothing$-satisfiable. Since we need only check a polynomial number of such systems in each recursion, the theorem follows. □

5 The Krom Case

In the case of propositional logic, satisfiability can be checked in polynomial time for so-called *Krom formulae* (i.e. CNF formulae with two literals or less per clause). Therefore, in this section we study the corresponding restriction on DSRs in order to check whether the restriction to Krom formulae yields a tractable satisfiability problem. The answer unfortunately turns out to be negative for both the case of $S_\varnothing$-models and S-models. Recall that the class of Krom formulae is not comparable to the class of Horn DSRs. The satisfiability problems are defined as follows:

KromDSRSat$_\varnothing$:
  Instance: A finite set Γ of Krom DSRs.
  Question: Does there exist an $S_\varnothing$-model of Γ?

KromDSRSat:
  Instance: A finite set Γ of Krom DSRs.
  Question: Does there exist an S-model of Γ?

Neither of these restrictions were investigated by Jonsson and Drakengren [1996].

We shall show that both of these problems are NP-complete.

Theorem 5.1 KromDSRSat is NP-complete.
Proof: DSRSat is in NP, by Lemma 3.9, so KromDSRSat is in NP.

We show hardness for NP by a polynomial-time reduction from Monotone 3sat, which is defined as follows.

Instance: Propositional logic formula $\alpha \equiv \bigwedge_i C_i$ over some set $\mathcal{P}$ of propositional symbols, where either $C_i \equiv p \vee q \vee r$ or $C_i \equiv \neg p \vee \neg q \vee \neg r$, for propositional symbols $p, q, r \in \mathcal{P}$.
Question: Is α satisfiable?

First, we show that the following problem is also NP-complete, by reducing Monotone 3sat to it:

Instance: Propositional logic formula $\alpha \equiv \bigwedge_i C_i$ over some set $\mathcal{P}$ of propositional symbols, where either $C_i \equiv p \vee q \vee r$ or $C_i \equiv \neg p \vee \neg q$, for propositional symbols $p, q, r \in \mathcal{P}$.
Question: Is α satisfiable?

Let $\alpha = \bigwedge_i C_i$ be an instance of Monotone 3sat, and construct an instance of the above problem as follows: For each $C_i \equiv \neg p \vee \neg q \vee \neg r$, introduce three new variables s_1, s_2 and s_3, and replace C_i by the following four clauses:

$$\neg p \vee \neg s_1, \quad \neg q \vee \neg s_2, \quad \neg r \vee \neg s_3, \quad s_1 \vee s_2 \vee s_3.$$

Denote the formula obtained by performing these replacements by α'. It is easy to see that α is satisfiable iff α' is, and that the reduction is polynomial.

Now, from an instance α of the above NP-complete problem, construct an instance Γ of KromDSRSat as follows. First, introduce a new set variable E, and for each propositional symbol p in α, introduce a set variable P. For each such variable P, add a relation $E \subseteq P$. We intend to model that P is false in a model iff $P = E$ holds in that model.

For each $C_i \equiv p \vee q \vee r$, add a relation $P \neq Q \vee R \neq E$, and for each $C_i \equiv \neg p \vee \neg r$, add a relation $P \subseteq E \vee R \subseteq E$. Collect all these formulae in Γ. It is easy to see that Γ has an S-interpretation iff α is satisfiable, and that the reduction is polynomial. The result follows. □

Theorem 5.2 KROMDSRSAT$_\varnothing$ is NP-complete.
Proof: Membership in NP follows from Lemma 3.9.

Let Γ be an instance of KROMDSRSAT, E a fresh variable, $\mathcal{P}$ the set of variables in Γ, and set

$$\Gamma' = \Gamma \cup \{E \,\mathsf{disj}\, E\} \cup \{P \neq E | P \in \mathcal{P}\}.$$

Now Γ' forces every variable not to take the value $\varnothing$ in any model, and thus Γ' has an $S_\varnothing$-model iff Γ has an S-model. The transformation is polynomial, and NP-completeness follows. □

We can now summarize the complexity results for the different restricted version of DSRs, from the proofs in this paper and the paper by Jonsson and Drakengren [1996]. The result can be found in Table 2.

	DSRSAT	DSRSAT$_\varnothing$
Atomic	P	P
2-Horn	P	P
Horn	P	NPC
Krom	NPC	NPC
General	NPC	NPC

Table 2: The complexity of reasoning about DSRs.

6 Application: Tractable Inference in Intuitionistic Logic

In this section we exploit a connection between topology and intuitionistic logic found by Tarski [1956], namely that entailment in intuitionistic logic corresponds to entailment using set equations and topological models, in order to restrict the general entailment problem of intuitionistic logic to one that is computable in polynomial time. By the fact that the general problem is PSPACE-complete [Statman, 1979; Hudelmaier, 1993], this represents an improvement, but of course at the cost of a more restricted language. To our knowledge the only such known tractable subclass is that of Nebel [Nebel, 1995], restricting queries of the form $\phi_1, \ldots, \phi_n \vdash_I \phi_{n+1}$ so that every ϕ_i is a member of the set $\mathcal{I}_b$, the set of *binary clauses*, containing formulae of the following kinds:

$$\neg x, \neg x \vee \neg y, \neg(x \wedge y), \neg x \vee y, x \rightarrow y,$$

$$x, x \vee y, x \wedge y, \neg x \wedge y, \neg x \wedge \neg y.$$

Later, we shall see that our class is incomparable with Nebel's.

First, we need a few concepts.

Definition 6.1 A *topological space* is a tuple $\langle \mathcal{U}, \mathcal{T} \rangle$, where $\{\varnothing, \mathcal{U}\} \subseteq \mathcal{T} \subseteq 2^{\mathcal{U}}$, and $\mathcal{T}$ is closed under finite intersections and arbitrary unions. The elements of $\mathcal{T}$ are called the *open sets*.

Given any set $Y \subseteq \mathcal{U}$, we define its *interior* $i(Y)$ as the union of all open sets contained in Y, i.e.

$$i(Y) = \bigcup_{t \in \mathcal{T}, t \subseteq Y} t.$$

Note that $i(Y)$ is always open, and that Y is open iff $i(Y) = Y$, so any topology is determined by $\langle \mathcal{U}, i \rangle$. $\square$

For a concise introduction to topology, see Arkhangelskii and Fedorchuk [1990].

Much of the following notation is taken from Bennett [1994].

Definition 6.2 Let $\mathcal{P}$ be an enumerable set of propositional symbols. Then α is a formula in propositional intuitionistic logic (or an $\mathcal{I}_0$ formula) over $\mathcal{P}$ iff α is a formula in propositional logic over $\mathcal{P}$. We use $\vdash_I$ to denote entailment in $\mathcal{I}_0$.

A *topological interpretation* of $\mathcal{I}_0$ is a tuple $\langle \mathcal{U}, \mathcal{T}, \mathcal{P}, d \rangle$, where $\langle \mathcal{U}, \mathcal{T} \rangle$ is a topological space (inducing the interior mapping i), $\mathcal{P}$ is an enumerable set of propositional symbols, and d is a denotation function assigning each $p \in \mathcal{P}$ a set in $\mathcal{T}$. The domain of d is extended to all $\mathcal{I}_0$ formulae formed from these variables by defining

- $d(\neg\alpha) = i(\overline{d(\alpha)})$,
- $d(\alpha \wedge \beta) = d(\alpha) \cap d(\beta)$,

- $d(\alpha \vee \beta) = d(\alpha) \cup d(\beta)$, and
- $d(\alpha \to \beta) = i(\overline{d(\alpha)} \cup d(\beta))$.

We use $\models_T$ to denote topological entailment, i.e. entailment between set-equations which may contain the interior operator i, using topological interpretations. □

Tarski [1956] has established the following connection between topology and intuitionistic logic.

Theorem 6.3 Let α be an $\mathcal{I}_0$ formula. Then $\vdash_I \alpha$ iff $\models_T d(\alpha) = \mathcal{U}$. □

The following generalisation was made by Bennett [1994].

Theorem 6.4 Let $\alpha_1, \ldots, \alpha_n, \beta$ be $\mathcal{I}_0$ formulae. Then

$$d(\alpha_1) = \mathcal{U} \wedge \ldots \wedge d(\alpha_n) = \mathcal{U} \models_T d(\beta) = \mathcal{U}$$

iff

$$\alpha_1, \ldots, \alpha_n \vdash_I \beta.$$

□

Next, a consequence of Theorem 6.4, needed in order to use a satisfiability algorithm for checking entailment.

Corollary 6.5 Let $\alpha_1, \ldots, \alpha_n, \beta$ be $\mathcal{I}_0$ formulae. Then

$$\alpha_1, \ldots, \alpha_n \vdash_I \beta$$

iff

$$d(\alpha_1) = \mathcal{U} \wedge \ldots \wedge d(\alpha_n) = \mathcal{U} \wedge d(\beta) \neq \mathcal{U}$$

does not have a topological model.

Proof:

$\Rightarrow$) Suppose $\alpha_1, \ldots, \alpha_n \vdash_I \beta$. By Theorem 6.4,

$$d(\alpha_1) = \mathcal{U} \wedge \ldots \wedge d(\alpha_n) = \mathcal{U} \models_T d(\beta) = \mathcal{U},$$

and thus

$$d(\alpha_1) = \mathcal{U} \wedge \ldots \wedge d(\alpha_n) = \mathcal{U} \wedge d(\beta) \neq \mathcal{U}$$

does not have a topological model.
$\Leftarrow$) Suppose

$$d(\alpha_1) = \mathcal{U} \wedge \ldots \wedge d(\alpha_n) = \mathcal{U} \wedge d(\beta) \neq \mathcal{U}$$

does not have a topological model. Then all topological interpretations that satisfy $d(\alpha_1) = \mathcal{U} \wedge \ldots \wedge d(\alpha_n) = \mathcal{U}$ have to satisfy also $d(\beta) = \mathcal{U}$; thus

$$d(\alpha_1) = \mathcal{U} \wedge \ldots \wedge d(\alpha_n) = \mathcal{U} \models_T d(\beta) = \mathcal{U},$$

and by Theorem 6.4, $\alpha_1, \ldots, \alpha_n \vdash_I \beta$. □

We shall use the 2-Horn-SAT algorithm for reasoning with the set equations imposed by the $\mathcal{I}_0$ formulae through the d function. For this, we need a few results.

Proposition 6.6 Let Γ be a set of $\mathcal{I}_0$ formulae and α_i, $i \in \{1 \ldots n\}$, $\mathcal{I}_0$ formulae. Then $\Gamma \vdash_I \alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n$ iff $\Gamma \vdash_I \alpha_i$ for every $i \in \{1 \ldots n\}$.
Proof: Easily obtained from the axioms of intuitionistic logic. For a list of these, see, for instance, Dragalin [1988]. □

Proposition 6.7 Let Γ be a set of $\mathcal{I}_0$ formulae and α, β $\mathcal{I}_0$ formulae. Then $\Gamma \vdash_I \alpha \rightarrow \beta$ iff $\Gamma, \alpha \vdash_I \beta$.
Proof: See e.g. Dragalin [1988]. □

The following theorem makes it possible to reason about sets alone, bypassing the requirement of topological spaces for the connection with intuitionistic logic stated in Theorem 6.3.

Theorem 6.8 Let Γ be a set 2-Horn DSRs where the set variable symbols (including $\mathcal{U}$ and $\varnothing$) are taken from a set $\mathcal{X}$, and construct the set Γ' as

$$\Gamma' = \Gamma \cup \{X \subseteq \mathcal{U} | X \in \mathcal{X}\} \cup \{\varnothing \,\mathsf{disj}\, \varnothing\}.$$

If there exists an $S_\varnothing$-model M for Γ', then there exists a topological space $\langle A, \mathcal{T}\rangle$ such that $M(\mathcal{U}) = A$, $M(\varnothing) = \varnothing$ and for each $X \in \mathcal{X}$, $M(\mathcal{X}) \in \mathcal{T}$.
Proof: Take an $S_\varnothing$-model M for Γ', set $\mathcal{B}' = \{M(X) | X \in \mathcal{X}\}$, and let $\mathcal{T}$ be obtained from $\mathcal{B}'$ by closing it under finite intersections and arbitrary unions ($\mathcal{B}'$ is a *subbasis* for $\mathcal{T}$). The relations in Γ' guarantee that every set in $\mathcal{T}$ is a subset of $M(\mathcal{U})$, so we can set $A = M(\mathcal{U})$. We also have $M(\varnothing) = \varnothing$, by the relation $\varnothing \,\mathsf{disj}\, \varnothing$ included in Γ'. □

The next result is crucial for translating formulae in intuitionistic formulae into set equations.

Proposition 6.9 First abbreviate "X is satisfiable iff Y is satisfiable" as "X siff Y", X and Y being set expressions, and "satisfiable" meaning having a topological model.

- $d(\alpha \wedge \beta) = \varnothing$ iff $d(\alpha) \,\mathsf{disj}\, d(\beta)$
- $d(\alpha_1 \vee \ldots \vee \alpha_n) = \varnothing$ iff $d(\alpha_1) = \varnothing \wedge \ldots \wedge d(\alpha_n) = \varnothing$
- $d(\neg\alpha) = \mathcal{U}$ iff $d(\alpha) = \varnothing$
- $d(\alpha_1 \wedge \ldots \wedge \alpha_n) = \mathcal{U}$ iff $d(\alpha_1) = \mathcal{U} \wedge \ldots \wedge d(\alpha_n) = \mathcal{U}$
- $d(\alpha \rightarrow \beta) = \mathcal{U}$ iff $d(\alpha) \subseteq d(\beta)$
- $d(\alpha \rightarrow \neg\beta) = \mathcal{U}$ iff $d(\alpha) \,\mathsf{disj}\, d(\beta)$
- $d(\alpha_1 \wedge \ldots \wedge \alpha_n) \neq \varnothing$ siff
  $X \neq \varnothing \wedge X \subseteq d(\alpha_1) \wedge \ldots \wedge X \subseteq d(\alpha_n)$,
  for a fresh variable X
- $d(\alpha_1 \vee \ldots \vee \alpha_n) \neq \varnothing$ iff $d(\alpha_1) \neq \varnothing \vee \ldots \vee d(\alpha_n) \neq \varnothing$
- $d(\neg\alpha) \neq \mathcal{U}$ iff $d(\alpha) \neq \varnothing$
- $d(\alpha_1 \vee \ldots \vee \alpha_n) \neq \mathcal{U}$ siff
  $X \neq \varnothing \wedge X \,\mathsf{disj}\, d(\alpha_1) \wedge \ldots \wedge X \,\mathsf{disj}\, d(\alpha_n)$,
  for a fresh variable X

Proof:

- $d(\alpha \wedge \beta) = \varnothing$ iff $d(\alpha) \cap d(\beta) = \varnothing$ iff $d(\alpha) \,\mathsf{disj}\, d(\beta)$
- $d(\alpha_1 \vee \ldots \vee \alpha_n) = \varnothing$ iff
  $d(\alpha_1) \cup \ldots \cup d(\alpha_n) = \varnothing$ iff $d(\alpha_1) = \varnothing \wedge \ldots \wedge d(\alpha_n) = \varnothing$
- $d(\neg\alpha) = \mathcal{U}$ iff $i(\overline{d(\alpha)}) = \mathcal{U}$ iff $\overline{d(\alpha)} = \mathcal{U}$ iff $d(\alpha) = \varnothing$
- $d(\alpha_1 \wedge \ldots \wedge \alpha_n) = \mathcal{U}$ iff $d(\alpha_1) \cap \ldots \cap d(\alpha_n) = \mathcal{U}$ iff
  $d(\alpha_1) = \mathcal{U} \wedge \ldots \wedge d(\alpha_n) = \mathcal{U}$
- $d(\alpha \rightarrow \beta) = \mathcal{U}$ iff $i(\overline{d(\alpha)} \cup d(\beta)) = \mathcal{U}$ iff
  $\overline{d(\alpha)} \cup d(\beta) = \mathcal{U}$ iff $d(\alpha) \subseteq d(\beta)$

- $d(\alpha \to \neg\beta) = \mathcal{U}$ iff $i(\overline{d(\alpha)} \cup d(\neg\beta)) = \mathcal{U}$ iff $\overline{d(\alpha)} \cup d(\neg\beta) = \mathcal{U}$ iff $d(\alpha) \subseteq d(\neg\beta)$ iff $d(\alpha) \subseteq i(\overline{d(\beta)})$ iff $d(\alpha) \subseteq \overline{d(\beta)}$ iff $d(\alpha) \,\mathsf{disj}\, d(\beta)$
- $d(\alpha_1 \wedge \ldots \wedge \alpha_n) \neq \varnothing$ iff $d(\alpha_1) \cap \ldots \cap d(\alpha_n) \neq \varnothing$ siff $X \neq \varnothing \wedge X \subseteq d(\alpha_1) \wedge \ldots \wedge X \subseteq d(\alpha_n)$, for a fresh variable X
- $d(\alpha_1 \vee \ldots \vee \alpha_n) \neq \varnothing$ iff $d(\alpha_1) \cup \ldots \cup d(\alpha_n) \neq \varnothing$ iff $d(\alpha_1) \neq \varnothing \vee \ldots \vee d(\alpha_n) \neq \varnothing$
- $d(\neg\alpha) \neq \mathcal{U}$ iff $i(\overline{d(\alpha)}) \neq \mathcal{U}$ iff $\overline{d(\alpha)} \neq \mathcal{U}$ iff $d(\alpha) \neq \varnothing$
- $d(\alpha_1 \vee \ldots \vee \alpha_n) \neq \mathcal{U}$ iff $d(\alpha_1) \cup \ldots \cup d(\alpha_n) \neq \mathcal{U}$ siff $X \neq \varnothing \wedge X \,\mathsf{disj}\, d(\alpha_1) \wedge \ldots \wedge X \,\mathsf{disj}\, d(\alpha_n)$, for a fresh variable X

□

The actual translation procedure is carried out using a rewrite relation:

Definition 6.10 Define the relations $\rightsquigarrow_i$ on set equations, by the following, where α, α_i and β are $\mathcal{I}_0$ formulae, and each new variable introduced is assumed to be unique.

$$\begin{array}{lll}
d(\alpha \wedge \beta) = \varnothing & \rightsquigarrow_1 & d(\alpha) \,\mathsf{disj}\, d(\beta) \\
d(\alpha_1 \vee \ldots \vee \alpha_n) = \varnothing & \rightsquigarrow_1 & d(\alpha_1) = \varnothing \wedge \ldots \wedge d(\alpha_n) = \varnothing \\
 & & \\
d(\neg\alpha) = \mathcal{U} & \rightsquigarrow_2 & d(\alpha) = \varnothing \\
d(\alpha_1 \wedge \ldots \wedge \alpha_n) = \mathcal{U} & \rightsquigarrow_2 & d(\alpha_1) = \mathcal{U} \wedge \ldots \wedge d(\alpha_n) = \mathcal{U} \\
d(\alpha \to \beta) = \mathcal{U} & \rightsquigarrow_2 & d(\alpha) \subseteq d(\beta) \\
d(\alpha \to \neg\beta) = \mathcal{U} & \rightsquigarrow_2 & d(\alpha) \,\mathsf{disj}\, d(\beta) \\
 & & \\
d(\alpha_1 \wedge \ldots \wedge \alpha_n) \neq \varnothing & \rightsquigarrow_3 & X \neq \varnothing \wedge X \subseteq d(\alpha_1) \wedge \ldots \wedge X \subseteq d(\alpha_n) \\
d(\alpha_1 \vee \ldots \vee \alpha_n) \neq \varnothing & \rightsquigarrow_3 & d(\alpha_1) \neq \varnothing \vee \ldots \vee d(\alpha_n) \neq \varnothing \\
 & & \\
d(\neg\alpha) \neq \mathcal{U} & \rightsquigarrow_4 & d(\alpha) \neq \varnothing \\
d(\alpha_1 \vee \ldots \vee \alpha_n) \neq \mathcal{U} & \rightsquigarrow_4 & X \neq \varnothing \wedge X \,\mathsf{disj}\, d(\alpha_1) \wedge \ldots \wedge X \,\mathsf{disj}\, d(\alpha_n)
\end{array}$$

Also define $\rightsquigarrow = \bigcup_i \rightsquigarrow_i$, and denote by $\Delta \rightsquigarrow^* \Delta'$ that there exists a (possibly empty) sequence $\Delta \rightsquigarrow \ldots \rightsquigarrow \Delta'$, that is *maximal*, i.e. it is impossible to further rewrite Δ' with $\rightsquigarrow$. □

Corollary 6.11 Let Δ be a set equation. Then there exists a set equation Δ' such that $\Delta \rightsquigarrow^* \Delta'$, and for no other $\Delta'' \neq \Delta'$, we have $\Delta \rightsquigarrow^* \Delta''$. Furthermore, the rewriting from Δ to Δ' can be done in polynomial time.
Proof: Since the $\rightsquigarrow_i$ relations can only rewrite each atomic relation in exactly one way, and in each step one redex is removed (a *redex* is a possibility to rewrite), Δ' will be uniquely determined by Δ. Obviously the rewriting terminates, and in polynomial time. □

Corollary 6.12 Let Δ be a set equation, and suppose that $\Delta \rightsquigarrow^* \Delta'$. Then Δ is satisfiable iff Δ' is satisfiable.
Proof: Directly from Proposition 6.9. □

Now, we can define the languages we need for specifying the tractable class for intuitionistic logic.

Definition 6.13 Let $\mathcal{P}$ be a set of propositional symbols. Define the sets $\mathcal{L}_i$, for $1 \leq i \leq 5$ in BNF as follows, letting p, q, p_j denote elements in $\mathcal{P}$, and α^i, α^i_j elements of $\mathcal{L}_i$.

$$\mathcal{L}_1 ::= \; p \mid p_1 \wedge p_2 \mid \alpha^1_1 \vee \ldots \vee \alpha^1_n$$

$$\mathcal{L}_2 ::= \; p \mid \neg\alpha^1 \mid \alpha^2_1 \wedge \ldots \wedge \alpha^2_n \mid p \rightarrow q \mid p \rightarrow \neg q$$

$$\mathcal{L}_3 ::= \; p \mid p_1 \wedge \ldots \wedge p_n \mid p_1 \vee \ldots \vee p_n$$

$$\mathcal{L}_4 ::= \; p \mid \neg\alpha^3 \mid p_1 \vee \ldots \vee p_n$$

$$\mathcal{L}_5 ::= \; \alpha^4_1 \wedge \ldots \wedge \alpha^4_n \mid (\alpha^2_1 \wedge \ldots \wedge \alpha^2_n) \rightarrow \alpha^5$$

□

We immediately see that the BNF definitions of the languages $\mathcal{L}_i$ follow the definitions of the rewrite relations $\rightsquigarrow_i$, except for $\mathcal{L}_5$. For example, taking $p \wedge q \in \mathcal{L}_1$, $d(p \wedge q) = \varnothing \rightsquigarrow_1 d(p) \,\mathsf{disj}\, d(q)$.

Proposition 6.14 Let $\alpha \in \mathcal{L}_2$ and $\beta \in \mathcal{L}_4$. Then $d(\alpha) = \mathcal{U} \rightsquigarrow^* \Delta$ and $d(\beta) \neq \mathcal{U} \rightsquigarrow^* \Delta'$, where Δ and Δ' are conjunctions of 2-Horn DSRs, satisfying that whenever an expression $d(\alpha)$ occurs in them, we have $\alpha \in \mathcal{P}$.
Proof: We first prove the result for $\alpha \in \mathcal{L}_2$ by induction, by also proving the stronger result that for $\beta \in \mathcal{L}_1$, the corresponding result holds for $d(\beta) = \varnothing$.

We begin with the basis steps:

- For $\beta \equiv p$, the result is trivial.
- For $\beta \equiv p_1 \wedge p_2$, the result follows by the definition of $\rightsquigarrow_1$.
- For $\alpha \equiv p$, the result is trivial.
- For $\alpha \equiv p \rightarrow q$, the result is immediate by the definition of $\rightsquigarrow_2$.
- For $\alpha \equiv p \rightarrow \neg q$, the result is immediate by the definition of $\rightsquigarrow_2$.

Next, the induction steps:

- For $\beta \equiv \alpha_1^1 \vee \ldots \vee \alpha_n^1 \in \mathcal{L}_1$, we get $d(\beta) = \varnothing \rightsquigarrow_1 d(\alpha_1^1) = \varnothing \wedge \ldots \wedge d(\alpha_n^1) = \varnothing$, and the result follows by induction.
- For $\alpha \equiv \neg\alpha^1$, we get $d(\neg\alpha^1) = \mathcal{U} \rightsquigarrow_2 d(\alpha^1) = \varnothing$, and the result follows by induction.
- For $\alpha \equiv \alpha_1^2 \wedge \ldots \wedge \alpha_n^2$, we get $d(\alpha_1^2 \wedge \ldots \wedge \alpha_n^2) = \mathcal{U} \rightsquigarrow_2 d(\alpha_1^2) = \mathcal{U} \wedge \ldots \wedge d(\alpha_1^2) = \mathcal{U}$, and the result follows by induction.

For $\beta \in \mathcal{L}_4$, no induction is needed. However, we first need to prove the result that for $\alpha \in \mathcal{L}_3$, the same holds for $d(\alpha) \neq \varnothing$.

- For $\alpha \equiv p$, the result is trivial.
- For $\alpha \equiv p_1 \wedge \ldots \wedge p_n$, the result follows from the definition of $\rightsquigarrow_3$.
- For $\alpha \equiv p_1 \vee \ldots \vee p_n$, the result follows from the definition of $\rightsquigarrow_3$.

Now the results for $\beta \in \mathcal{L}_4$:

- For $\beta \equiv p$, the result is trivial.
- For $\beta \equiv p_1 \vee \ldots \vee p_n$, the result follows from the definition of $\rightsquigarrow_4$.
- For $\beta \equiv \neg\alpha^3$, we get $d(\neg\alpha^3) \neq \mathcal{U} \rightsquigarrow_4 d(\alpha^3) \neq \varnothing$, and the result follows by the property of $\mathcal{L}_3$.

$\square$

Algorithm 6.15 (Entailment(Γ, β)**)**

input A finite set $\Gamma = \{\alpha_1, \ldots, \alpha_n\} \subseteq \mathcal{L}_2$, $\beta \in \mathcal{L}_5$, and .

```
 1  if n > 1 then
 2     for each β_i
 3        if not Entailment(Γ, β_i) then
 4           reject
 5     accept
 6  elsif β ≡ γ → δ then
 7     Entailment(Γ ∪ {γ}, δ)
 8  else
 9     Compute d(⋀Γ) = U ∧ d(β) ≠ U ⇝* Δ
10     2-Horn-SAT({Δ} ∪ {d(p) ⊆ U | p ∈ P} ∪ {∅ disj ∅})
11  endif
```

□

Theorem 6.16 Let $\Gamma \subseteq \mathcal{L}_2$ and $\beta \in \mathcal{L}_5$. Then Algorithm 6.15 decides whether $\Gamma \vdash_I \beta$, and furthermore, it does so in polynomial time.

Proof: First, if $\beta \equiv \bigwedge_i \beta_i$, then by Proposition 6.6 it is necessary and sufficient to check entailment for the β_is separately, which is done in lines 1-5. Then, as long as $\beta \equiv \gamma \to \delta$, we may move γ to the left of $\vdash_I$, and keep δ, by Proposition 6.7, which is done in lines 6-7. Then we are left at line 9 with a problem of the form

$$\Gamma \vdash_I \beta,$$

where $\Gamma \subseteq \mathcal{L}_2$ and $\beta \in \mathcal{L}_4$, by the construction of $\mathcal{L}_5$. Since $\mathcal{L}_2$ is closed under finite conjunction, $\bigwedge \Gamma \in \mathcal{L}_2$. Thus, by Proposition 6.14 we can rewrite the expression

$$d(\bigwedge \Gamma) = \mathcal{U} \wedge d(\beta) \neq \mathcal{U}$$

using $\rightsquigarrow^*$ into a set expression Δ where all occurrences of $d(\alpha)$ satisfy $\alpha \in \mathcal{P}$. Furthermore, this expression is a conjunction of 2-Horn DSRs in the variables $d(p)$ (since the value of propositional symbols in a model are independent, we can consider the expressions $d(p)$ to be variables), and it is satisfiable iff the the original expression is satisfiable, by Corollary 6.12. By Corollary 6.5, Δ is satisfiable by a topological model iff $\Gamma \vdash_I \beta$.

Let $\Omega = \{\Delta\} \cup \{d(p) \subseteq \mathcal{U} | p \in \mathcal{P}\} \cup \{\varnothing \ \mathsf{disj} \ \varnothing\}$. If now Δ has a topological model, then it is certainly an $S_\varnothing$-model of Ω. Conversely, if Ω has an $S_\varnothing$-model, then by Theorem 6.8, Δ has a topological model, and correctness follows.

Polynomiality follows from the polynomiality of 2-Horn-SAT, since the rewritings in line 9 using $\leadsto^*$ are clearly polynomial. $\Box$

Note that this class does not subsume that of Nebel [1995], since, for example, formulae on the form $\neg x \vee \neg y$ are not included here. However, Nebel's class does not subsume this class either, since e.g. $(x \rightarrow y) \rightarrow (w \rightarrow z)$ is not included therein.

7 Discussion

Having identified a tractable fragment of intuitionistic logic, several interesting questions arise. Nebel [1995] uses his tractable class in order to obtain a tractable class for spatial reasoning, in the so-called RCC-8 spatial algebra [Randell and Cohn, 1989; Randell *et al.*, 1992]. This class was later extended by Renz [1996] to a maximal tractable subclass of that spatial algebra. Thus, since this class is incomparable to Nebel's, is it possible to use it to obtain other tractable subclasses of RCC-8, by reducing their satisfiability problem to that of intuitionistic logic? For the case of the RCC-5 spatial algebra, all possible cases of tractable subclasses have already been characterised [Renz, 1996; Jonsson and Drakengren, 1997].

Another relevant question is whether we can use our tractable class of set constraints for tractable inference in other logical systems. For instance, Renz [1996] uses classes of modal logics in order to prove classes of the RCC-5 and RCC-8 spatial algebras tractable, so there is obviously a connection between these formalisms, and our reasoning about sets. It would thus be interesting to investigate whether we can find tractable classes of these logics, and related formalisms.

8 Conclusions

We have found a new class of set constraints, the *2-Horn DSRs*, for which reasoning is tractable (i.e. polynomial). This class is used for constructing a new class in intuitionistic logic, where tractable inference can be performed. In order to investigate the borderline between tractable and intractable reasoning with set constraints, we prove that a class of set constraints previously proved tractable, the *Horn DSRs*, becomes intractable if the empty set is allowed as values for set variables. In addition, we prove that the restriction to *Krom formulae*, which is known to be polynomial for propositional logic, is still NP-complete for set constraints, whether or not the empty set is allowed for set variables.

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Linköping Studies in Science and Technology

Dissertation No. 498

Algorithms and Complexity for Temporal and Spatial Formalisms

Thomas Drakengren

Akademisk avhandling

som för avläggande av filosofie doktorsexamen vid Linköpings universitet kommer att offentligt försvaras i Hörsal Planck, hus F, Linköpings universitet, tisdagen den 28 oktober 1997, kl 16.00.

ABSTRACT

The problem of computing with temporal information was early recognised within the area of artificial intelligence, most notably the temporal *interval algebra* by Allen has become a widely used formalism for representing and computing with qualitative knowledge about relations between temporal intervals. However, the computational properties of the algebra and related formalisms are known to be bad: most problems (like *satisfiability*) are NP-hard. This thesis contributes to finding restrictions (as weak as possible) on Allen's algebra and related temporal formalisms (the *point-interval algebra* and extensions of Allen's algebra for metric time) for which the satisfiability problem can be computed in polynomial time.

Another research area utilising temporal information is that of *reasoning about action*, which treats the problem of drawing conclusions based on the knowledge about actions having been performed at certain time points (this amounts to solving the infamous *frame problem*). One paper of this thesis attacks the computational side of this problem; one that has not been treated in the literature (research in the area has focused on *modelling* only). A nontrivial class of problems for which satisfiability is a polynomial-time problem is isolated, being able to express phenomena such as concurrency, conditional actions and continuous time.

Similar to temporal reasoning is the field of *spatial reasoning*, where spatial instead of temporal objects are the field of study. In two papers, the formalism *RCC-5* for spatial reasoning, very similar to Allen's algebra, is analysed with respect to tractable subclasses, using techniques from temporal reasoning.

Finally, as a spin-off effect from the papers on spatial reasoning, a technique employed therein is used for finding a class of *intuitionistic logic* for which computing inference is tractable.

This work was partially supported by the Swedish Research Council for the Engineering Sciences (TFR), grant 93-270.

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ISBN 91-7219-019-1 ISSN 0345-7524