Towards a Complete Classification of Tractability in Allen's Algebra

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Abstract

We characterise the set of subalgebras of Allen's algebra which have a tractable satisfiability problem, and in addition contain certain basic relations. The conclusion is that no tractable subalgebra that is not known in the literature can contain more than the three basic relations $(\equiv)$, (b) and $(b^{\smile})$, where $b \in \{\mathsf{d}, \mathsf{o}, \mathsf{s}, \mathsf{f}\}$. This means that concerning algebras for specifying *complete knowledge* about temporal information, there is no hope of finding yet unknown classes with much expressivity. Furthermore, we show that there are exactly two maximal tractable algebras which contain the relation $(\prec \succ)$. Both of these algebras can express the notion of *sequentiality*; thus we have a complete characterisation of tractable inference using that notion.

1 Introduction

This paper improves on known results about algorithms for the problem of reasoning about temporal constraints. Such reasoning is an important task in many areas of AI and elsewehere, such as planning [Allen, 1991], natural language processing [Song and Cohen, 1988], time serialization in archeology [Golumbic and Shamir, 1993] and more, and there are several frameworks for formalising such problems, according to different needs. Among the most frequently used ones are the *point algebra* [van Beek and Cohen, 1990], used for expressing qualitative relations between time points, the *point-interval* algebra [Vilain, 1982] for expressing qualitative relations between time points and time intervals, and the famous interval algebra of Allen [1983] for expressing qualitative relations between time intervals. There are also combinations of these and extensions to handle also metric time, such as Meiri's framework [Meiri, 1991], and the works of Kautz and Ladkin [1991], Gerevini *et al.* [1993], Dechter *et al.* [1991], Jonsson and Bäckström [1996] and Drakengren and Jonsson [1997]. However, it was early proved that the reasoning problem for these formalisms is very hard; e.g. reasoning in Allen's interval algebra is NP-complete [Vilain and Kautz, 1986], and NP-hardness carries over to more expressive formalisms.

These computational problems have motivated the search for various tractable fragments of the temporal formalisms, where reasoning can be guaranteed to be reasonably efficient. In particular, several subclasses of Allen's algebra have been reported tractable (we assume $P \neq NP$) [van Beek and Cohen, 1990; Golumbic and Shamir, 1993; Nebel and Bürckert, 1995; Drakengren and Jonsson, 1996; 1997]. However, in view of the large number of possible subclasses of Allen's algebra (the algebra contains 8192 relations, leading to $2^{8192} \approx 10^{2466}$ subclasses), such results are in danger of appearing *ad hoc*. As a first reaction to this, research has recently focused on identifying *maximal* tractable subclasses; i.e. classes which cannot be extended without losing tractability. This direction is clearly more systematic, since any tractable subclass is included in a maximal tractable one. The first such algebra was identified by Nebel and Bürckert [1995], soon to be followed by Drakengren and Jonsson [1996, 1997], resulting in eighteen known maximal algebras, subsuming all algebras previously known to be tractable. Still, however, this is a very small number compared to the total number of possible subclasses.

Due to this apparent lack of systematicity, techniques have recently been developed allowing *full* classifications of tractability, in particular for the point-interval algebra [Jonsson *et al.*, 1996], but also for the RCC-5 algebra for *spatial* reasoning [Jonsson and Drakengren, 1997]. A full classification of tractability for an algebra means that we identify the *complete* set of tractable subclasses in the algebra. Despite the success for the point-interval algebra and the RCC-5 algebra, the corresponding task for Allen's algebra poses a problem more difficult by several orders of magnitude: the number of subclasses in these algebras is only $2^{32} \approx 4.3 \cdot 10^9$. In principle, all these can be enumerated on a computer, but this is certainly not the case with the Allen algebra.

In this context, this paper presents a significant step towards a full classification of tractability in Allen's algebra. We show that any algebra that is yet to be found can contain at most three basic relations: $(\equiv)$, (b) and $(b^{\smile})$, for $b \in \{\mathsf{d}, \mathsf{o}, \mathsf{s}, \mathsf{f}\}$. This means that in order to

Basic relation		Example	Endpoints
x before y	$\prec$	xxx	$x^+ < y^-$
y after x	$\succ$	yyy	
x meets y	m	xxxx	$x^+ = y^-$
y met-by x	$\mathsf{m}^\smile$	yyyy	
x overlaps y	o	xxxx	$x^- < y^- < x^+$,
y overl.-by x	$\mathsf{o}^\smile$	yyyy	$x^+ < y^+$
x during y	d	xxx	$x^- > y^-$,
y includes x	$\mathsf{d}^\smile$	yyyyyyy	$x^+ < y^+$
x starts y	s	xxx	$x^- = y^-$,
y started by x	$\mathsf{s}^\smile$	yyyyyyy	$x^+ < y^+$
x finishes y	f	xxx	$x^+ = y^+$,
y finished by x	$\mathsf{f}^\smile$	yyyyyyy	$x^- > y^-$
x equals y	$\equiv$	xxxx	$x^- = y^-$,
		yyyy	$x^+ = y^+$

Table 1: The thirteen basic relations.

specify complete temporal knowledge, we cannot hope to find more expressive algebras than those already known. Furthermore, we show that there are exactly two maximal tractable algebras which can express the important notion of *sequentiality* [Sandewall, 1994].

Finally, note that the main results of this paper are proved using exhaustive search by computers. Naturally, such proofs cannot be reproduced in a paper, but we encourage researchers in the field to repeat our proofs. All software used in the paper can be obtained from the authors.

The structure of the paper follows. First we present Allen's algebra in Section 2, after which the classification results follow. A discussion concludes the paper. The more complicated proofs are collected in an appendix.

2 Allen's Algebra

Allen's interval algebra [Allen, 1983] is based on the notion of *relations between pairs of intervals*. An interval x is represented as a tuple $\langle x^-, x^+\rangle$ of real numbers with $x^- < x^+$, denoting the left and right endpoints of the interval, respectively, and relations between intervals are composed as disjunctions of *basic interval relations*, which are those in Table 1 (denoted **B**). Such disjunctions are represented as *sets* of basic relations, but using a notation such that e.g. the disjunction of the basic intervals $\prec$, m and $\mathsf{f}^\smile$ is written $(\prec\ \mathsf{m}\,\mathsf{f}^\smile)$. Thus, we have that $(\prec\ \mathsf{f}^\smile) \subseteq (\prec\ \mathsf{m}\,\mathsf{f}^\smile)$. Sometimes, the disjunction of *all* basic relations is written $\top$, and the empty relation is written $\perp$ (this also used for relations between interval endpoints, denoting "always satisfiable" and "unsatisfiable", respectively). The algebra is provided with the operations of *converse, intersection and composition* on intervals, but we shall need only the converse operation. The converse operation takes an interval relation i to its converse $i^\smile$, obtained by inverting each basic relation in i, i.e., exchanging x and y in the endpoint relations of Table 1.

By the fact that there are thirteen basic relations, we get $2^{13} = 8192$ possible relations between intervals in the full algebra. We denote the set of all interval relations by $\mathcal{A}$. Subclasses of the full algebra are obtained by considering subsets of $\mathcal{A}$. There are $2^{8192} \approx 10^{2466}$ such subclasses.

Although there are several computational problems associated with Allen's interval algebra, this paper focuses on the problem of *satisfiability* of a set of interval variables with relations between them, i.e. deciding whether there exists an assignment of intervals on the real line for the interval variables, such that all of the relations between the intervals are satisfied. We define this as follows.

Definition 2.1 ($\mathcal{A}$-SAT($\mathcal{I}$)) Let $\mathcal{I}$ be a set of interval relations. An instance of $\mathcal{A}$-SAT($\mathcal{I}$) is a labelled directed graph $S = \langle V, E\rangle$, where the nodes in V are interval variables and E is a subset of $V \times \mathcal{I} \times V$. A labelled edge $\langle u, r, v\rangle \in E$ means that u and v are related by r.

A function M taking an interval variable v to its interval representation $M(v) = \langle x^-, x^+\rangle$ with $x^- < x^+$, $x^-, x^+ \in \mathbb{R}$, is said to be an *interpretation* of S.

An instance $\langle V, E\rangle$ is said to be *satisfiable* iff there exists an interpretation M such that for each $\langle u, r, v\rangle \in E$, $M(u) r M(v)$ holds, i.e. the endpoint relations required by r (see Table 1) are satisfied by the assignments of u and v. Then M is said to be a *model* of $\langle V, E\rangle$.

We refer to the *size* of an instance $\langle V, E\rangle$ as $|V| + |E|$. $\square$

For $\mathcal{A}$, we have the following result.

Proposition 2.2 $\mathcal{A}$-SAT($\mathcal{A}$) is NP-complete.
Proof: See Vilain and Kautz [1986]. $\square$

Next, we introduce Nebel and Bürckert's [1995] *closure* operation, here denoted $\mathcal{C}_{\mathcal{A}}(\cdot)$, which transforms a given subclass of $\mathcal{A}$ to one that is polynomially equivalent to the original subclass wrt. satisfiability.

Definition 2.3 (Closure) Let $\mathcal{S} \subseteq \mathcal{A}$. Then we denote by $\mathcal{C}_{\mathcal{A}}(\mathcal{S})$ the $\mathcal{A}$-*closure* of $\mathcal{S}$, defined as the least subalgebra containing $\mathcal{S}$ and which is closed under converse, intersection and composition. $\square$

Closures can be computed using Nebel and Bürckert's software [1993].

The key result for extrapolating complexity results is the following.

Proposition 2.4 Let $\mathcal{S} \subseteq \mathcal{A}$. Then $\mathcal{A}$-SAT($\mathcal{S}$) is polynomial iff $\mathcal{A}$-SAT($\mathcal{C}_{\mathcal{A}}(\mathcal{S})$) is, and $\mathcal{A}$-SAT($\mathcal{S}$) is NP-complete iff $\mathcal{A}$-SAT($\mathcal{C}_{\mathcal{A}}(\mathcal{S})$) is.
Proof: See Nebel and Bürckert [1995]. $\square$

$\mathcal{A}$-SAT is sometimes defined such that for each pair of objects (e.g. time intervals), we have exactly one relation (cf. Golumbic and Shamir [1993]). In this way, the reduction needed for Proposition 2.4 would fail, since intervals which are added are not always related.

3 Classification of $\mathcal{A}$

This section contains the parts of the classification.

3.1 Intractable Subclasses

In order to provide the classification, we need to find more NP-complete subclasses of $\mathcal{A}$ than those previously known. Our main tools for proving intractability are the following NP-complete subclasses of $\mathcal{A}$.

Definition 3.1 (Subclasses $\mathcal{N}_i$, relation R, sets $\mathcal{A}_0$, $\mathcal{A}_4$ and $\mathcal{A}_{NP}$) First define the auxiliary set A by $A = \{(\prec\ \mathsf{d}^{\smile}\ \mathsf{o}\ \mathsf{m}\ \mathsf{f}^{\smile}), (\prec\ \mathsf{d}\ \mathsf{o}\ \mathsf{m}\ \mathsf{s})\}$. Define the following sets.

$$\mathcal{N}_1 = A \cup \{(\mathsf{d}\ \mathsf{d}^{\smile}\ \mathsf{o}^{\smile}\ \mathsf{s}^{\smile}\ \mathsf{f})\},$$

$$\mathcal{N}_2 = A \cup \{(\mathsf{d}^{\smile}\ \mathsf{o}\ \mathsf{o}^{\smile}\ \mathsf{s}^{\smile}\ \mathsf{f}^{\smile})\},$$

$$\mathcal{N}_3 = \{(\prec\ \succ), (\mathsf{o}\ \mathsf{o}^{\smile})\},$$

$$\mathcal{N}_4 = \{(\prec\ \succ), (\mathsf{o}\ \mathsf{o}^{\smile}\ \mathsf{m}\ \mathsf{m}^{\smile})\},$$

$$\mathcal{N}_5 = \{(\mathsf{m}\ \mathsf{m}^{\smile}), (\prec\ \succ\ \mathsf{s}\ \mathsf{s}^{\smile}\ \mathsf{f}\ \mathsf{f}^{\smile})\}.$$

Define the relations R and R' by $R = (\mathsf{d}\ \mathsf{d}^{\smile}\ \mathsf{o}\ \mathsf{o}^{\smile})$ and $R' = (\equiv\ \mathsf{m}\ \mathsf{m}^{\smile}\ \mathsf{s}\ \mathsf{s}^{\smile}\ \mathsf{f}\ \mathsf{f}^{\smile})$, and set $\mathcal{A}_{NP}$ to be the union of the following sets:

$$\{\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4, \mathcal{N}_5\},$$

$$\mathcal{A}_0 = \{\{(\prec\ \succ), r\} | R \subseteq r \subseteq R \cup R'\},$$

$$\mathcal{A}_4 = \{\{(\prec\ \succ), r\} | R \cup (\prec) \subseteq r \subseteq R \cup (\prec) \cup R'\}.$$

□

Proposition 3.2 $\mathcal{A}$-SAT($\mathcal{S}$) is NP-complete for all $\mathcal{S} \in \mathcal{A}_{NP}$.

Proof: For $\mathcal{N}_1$ and $\mathcal{N}_2$, see Nebel and Bürckert [1995]. The remaining cases are proved in Theorem A.9 ($\mathcal{A}_0 \cup \mathcal{A}_4$) and Theorem A.12 ($\mathcal{N}_3$, $\mathcal{N}_4$ and $\mathcal{N}_5$) in the appendix. □

3.2 Tractable Algebras

Next we define what are the polynomial algebras involved in the classification.

Definition 3.3 ($bas(A)$, polynomial algebras) Let $bas(A)$ for $A \subseteq \mathcal{A}$ be the set of basic relations contained in A. Also let $\mathcal{H}$ denote the ORD-Horn algebra by Nebel and Bürckert [1995] and $\mathcal{S}_\prec$, $\mathcal{S}_\mathsf{d}$, $\mathcal{S}_\mathsf{o}$, $\mathcal{E}_\prec$, $\mathcal{E}_\mathsf{d}$ and $\mathcal{E}_\mathsf{o}$ the maximal tractable algebras of Drakengren and Jonsson [1997], where $\mathcal{S}_r$ denotes the unique *starting point algebra* containing the basic relation (r), and $\mathcal{E}_r$ the unique *ending point algebra* containing the basic relation (r). □

The following facts about the algebras shall be needed in the classification.

Proposition 3.4 $\mathcal{H}$, $\mathcal{S}_r$ and $\mathcal{E}_r$ are maximal tractable subclasses of $\mathcal{A}$, i.e. it is impossible to extend them without losing tractability. Furthermore, any tractable subclass $A \subseteq \mathcal{A}$ with $\mathbf{B} \subseteq A$ satisfies $A \subseteq \mathcal{H}$. Also, $bas(\mathcal{H}) = \mathbf{B}$, $bas(\mathcal{S}_r) = \{\equiv, r, r^{\smile}, \mathsf{s}, \mathsf{s}^{\smile}\}$, and $bas(\mathcal{E}_r) = \{\equiv, r, r^{\smile}, \mathsf{f}, \mathsf{f}^{\smile}\}$ for all $r \in \{\prec, \mathsf{d}, \mathsf{o}\}$.

Proof: The proofs for $\mathcal{H}$ can be found in [Nebel and Bürckert, 1995], and those for $\mathcal{S}_r$ and $\mathcal{E}_r$ in [Drakengren and Jonsson, 1997]. □

In order to define the subject of our classification, define $\mathcal{T}$ to be the set of maximal tractable subalgebras of $\mathcal{A}$ not included in $\mathcal{H}$, $\mathcal{S}_r$ or $\mathcal{E}_r$, for any $r \in \{\prec, \mathsf{d}, \mathsf{o}\}$. Note that it is sufficient to restrict the attention to maximal tractable algebras, since any tractable subset can be extended to such an algebra. Also note that some of the algebras known from the literature (those of Drakengren and Jonsson [1996, 1997]) are included in $\mathcal{T}$, but this will not affect the classification, since these all contain three basic relations or less.

3.3 The Classification

We start by stating the main theorem of the paper, from which the classification results will follow. Since this kind of result has already been needed at least twice in the literature [Jonsson *et al.*, 1996; Jonsson and Drakengren, 1997], we take the opportunity to abstract it in order to make future classification results easier to state.

Theorem 3.5 Let $\mathcal{R}$ be a set equipped with an operation $\mathcal{C}_\mathcal{R}(R)$ on sets $R \subseteq \mathcal{R}$, and for each set $R \subseteq \mathcal{R}$ a problem $\mathcal{R}$-SAT(R), satisfying the following:

- If $\mathcal{R}$-SAT($\mathcal{C}_\mathcal{R}(R)$) is NP-complete, then $\mathcal{R}$-SAT(R) is NP-complete
- If $\mathcal{R}$-SAT(R) is NP-complete, then $\mathcal{R}$-SAT(S) is NP-complete for all $S \supseteq R$
- If $\mathcal{R}$-SAT(R) is polynomial, then $\mathcal{R}$-SAT(S) is polynomial for all $S \subseteq R$.

Let $\mathcal{R}_P, \mathcal{R}_{NP} \subseteq 2^\mathcal{R}$ and $\mathcal{B} \subseteq \mathcal{R}$, such that $\mathcal{R}$-SAT(X) is polynomial for each $X \in \mathcal{R}_P$, each $X \in \mathcal{R}_P$ satisfies $\mathcal{B} \subseteq X$, and $\mathcal{R}$-SAT(D) is NP-complete for each $D \in \mathcal{R}_{NP}$.

Then if each set $T \subseteq \mathcal{R}$ with $|T| \leq |\mathcal{R}_P|$ satisfies either that T is a subset of some set in $\mathcal{R}_P$, or that $D \subseteq \mathcal{C}_\mathcal{R}(T \cup \mathcal{B})$ for some $D \in \mathcal{R}_{NP}$, then for any $\mathcal{S}$ with $\mathcal{B} \subseteq \mathcal{S}$, $\mathcal{R}$-SAT($\mathcal{S}$) is polynomial iff $\mathcal{S}$ is a subset of some set in $\mathcal{R}_P$. Otherwise $\mathcal{R}$-SAT($\mathcal{S}$) is NP-complete.

Proof:

$\Leftarrow$) For each $R \in \mathcal{R}_P$, $\mathcal{R}$-SAT(R) is polynomial by definition, and so are subsets of R.

$\Rightarrow$) Consider a set $\mathcal{S} \subseteq \mathcal{A}$ with $\mathcal{B} \subseteq \mathcal{S}$, $\mathcal{S}$ not being a subset of any set in $\mathcal{R}_P$. For each set C in $\mathcal{R}_P$, choose an element x such that $x \in \mathcal{S}$ and $x \notin C$. This can always be done since $\mathcal{S} \not\subseteq C$. Let X be the set of these elements. By the construction of X, $|X| \leq |\mathcal{R}_P|$. But then, by the condition of the theorem, either X is a subset of some set in $\mathcal{R}_P$, or $D \subseteq \mathcal{C}_\mathcal{R}(X \cup \mathcal{B})$ for some $D \in \mathcal{R}_{NP}$. But the former case cannot hold by the construction of X; thus $\mathcal{R}$-SAT($\mathcal{C}_\mathcal{R}(X \cup \mathcal{B})$) is NP-complete. It follows that $\mathcal{R}$-SAT($X \cup \mathcal{B}$) is NP-complete, and since $X \cup \mathcal{B} \subseteq \mathcal{S}$, that $\mathcal{R}$-SAT($\mathcal{S}$) is NP-complete. The result follows. □

We now proceed gradually with the classification by excluding certain combinations of basic relations. Note that the three conditions making Theorem 3.5 applicable always hold for Allen's algebra. Also note that any algebra has to contain an odd number of basic relations, since algebras are closed under the converse operation, and $(\equiv)$ is always included.

The following result is similar to one of Drakengren and Jonsson [1997].

Proposition 3.6 Let $A \subseteq \mathcal{A}$. If $A \subseteq \mathcal{A}$ and $(\mathsf{m}) \in A$, then either $A \subseteq \mathcal{H}$ or $\mathcal{A}$-SAT(A) is NP-complete.
Proof: It can easily be verified that $\mathbf{B} \subseteq \mathcal{C}_{\mathcal{A}}(\{(\mathsf{m})\})$ (use the `aclose` utility by Nebel and Bürckert [1993]), and the result follows by Proposition 3.4. □

Thus, $A \in \mathcal{T} \Rightarrow bas(A) \leq 11$.

Now for the first application of the quite abstract Theorem 3.5.

Proposition 3.7 Let $A \subseteq \mathcal{A}$. If $(\prec) \in bas(A)$, then either $A \subseteq \mathcal{H}$, $A \subseteq \mathcal{S}_{\prec}$, $A \subseteq \mathcal{E}_{\prec}$, or $\mathcal{A}$-SAT(A) is NP-complete.
Proof: First choose $\mathcal{R} = \mathcal{A}$, $\mathcal{R}_P = \{\mathcal{H}, \mathcal{S}_{\prec}, \mathcal{E}_{\prec}\}$, $\mathcal{R}_{NP} = \mathcal{A}_{NP}$ and $\mathcal{B} = \{(\prec)\}$. Then enumerate each set $T \subseteq \mathcal{A}$ with $|T| \leq |\mathcal{R}_P| = 3$ and test if for each T, either $T \subseteq \mathcal{H}$, $T \subseteq \mathcal{S}_{\prec}$, $T \subseteq \mathcal{E}_{\prec}$, or $D \subseteq \mathcal{C}_{\mathcal{A}}(T \cup \mathcal{B})$ for some $D \in \mathcal{A}_{NP}$. There are $\sum_{i=0}^{3} \binom{8192}{i} \approx 9.2 \cdot 10^{10}$ such subsets. The test succeeds for all T, and the result follows. □

The subsets were enumerated on several Sun SPARC 10 stations in parallel, taking approximately 40 CPU weeks.

By this result, $A \in \mathcal{T} \Rightarrow bas(A) \leq 9$. The basic relations remaining to check are those in $Z = \{\mathsf{d}, \mathsf{d}^{\smile}, \mathsf{o}, \mathsf{o}^{\smile}, \mathsf{s}, \mathsf{s}^{\smile}, \mathsf{f}, \mathsf{f}^{\smile}\}$. If we can show that for any $r_1, r_2 \in Z$ with $r_1 \neq r_2$ and $r_1{}^{\smile} \neq r_2$, if for some $A \in \mathcal{T}$, $\{r_1, r_2\} \subseteq bas(A)$, then $A \subseteq \mathcal{H}$, $A \subseteq \mathcal{S}_r$ or $A \subseteq \mathcal{E}_r$ for some r, then we could conclude that $A \in \mathcal{T} \Rightarrow bas(A) \leq 3$, which is the goal of the paper. The following results will prove this.

Proposition 3.8 Let $A \subseteq \mathcal{A}$. For $W = \{(\mathsf{d}), (\mathsf{o})\}$ or $W = \{(\mathsf{s}), (\mathsf{f})\}$, if $W \subseteq bas(A)$, then either $A \subseteq \mathcal{H}$, or $\mathcal{A}$-SAT(A) is NP-complete.
Proof: First choose $\mathcal{R} = \mathcal{A}$, $\mathcal{R}_P = \{\mathcal{H}\}$, $\mathcal{R}_{NP} = \mathcal{A}_{NP}$ and $\mathcal{B} = W$. Then enumerate each set $T \subseteq \mathcal{A}$ with $|T| \leq |\mathcal{R}_P| = 1$ and test if for each T, either $T \subseteq \mathcal{H}$ or $D \subseteq \mathcal{C}_{\mathcal{A}}(T \cup \mathcal{B})$ for some $D \in \mathcal{A}_{NP}$. There are 8193 such subsets, regardless of W. The test succeeds for all T, and the result follows from Theorem 3.5. □

Proposition 3.9 Let $A \subseteq \mathcal{A}$. If $\{(\mathsf{s}), (r)\} \subseteq bas(A)$ for $r \in \{\mathsf{d}, \mathsf{o}\}$, then either $A \subseteq \mathcal{H}$, $A \subseteq \mathcal{S}_r$, or $\mathcal{A}$-SAT(A) is NP-complete.
Proof: First choose $\mathcal{R} = \mathcal{A}$, $\mathcal{R}_P = \{\mathcal{H}, \mathcal{S}_r\}$, $\mathcal{R}_{NP} = \mathcal{A}_{NP}$ and $\mathcal{B} = \{(\mathsf{s}), (r)\}$. Then enumerate each set $T \subseteq \mathcal{A}$ with $|T| \leq |\mathcal{R}_P| = 2$ and test if for each T, either $T \subseteq \mathcal{H}$, $T \subseteq \mathcal{S}_r$, or $D \subseteq \mathcal{C}_{\mathcal{A}}(T \cup \mathcal{B})$ for some $D \in \mathcal{A}_{NP}$. There are $\approx 3.4 \cdot 10^7$ such subsets. The test succeeds for all T, and the result follows from Theorem 3.5. □

The cases with $\{(\mathsf{f}), (\mathsf{d})\}$ and $\{(\mathsf{f}), (\mathsf{o})\}$ follow by symmetry from Proposition 3.9, using $\mathcal{E}_r$ instead of $\mathcal{S}_r$. We can thus conclude that $A \in \mathcal{T} \Rightarrow bas(A) \leq 3$, and that algebras in $\mathcal{T}$ can only contain basic relations in $\{\equiv, \mathsf{d}, \mathsf{o}, \mathsf{s}, \mathsf{f}\}$ which is the main result of the paper.

We conclude by a classification of all algebras containing the relation $(\prec \succ)$, needed for expressing the notion of *sequentiality*. This notion is important in many AI contexts, such as planning and reasoning about action [Sandewall, 1994], where actions are often assumed to come in sequence.

Proposition 3.10 Let $A \subseteq \mathcal{A}$. If $(\prec \succ) \in A$, then either $A \subseteq \mathcal{S}_{\prec}$, $A \subseteq \mathcal{E}_{\prec}$, or $\mathcal{A}$-SAT(A) is NP-complete.
Proof: First choose $\mathcal{R} = \mathcal{A}$, $\mathcal{R}_P = \{\mathcal{S}_{\prec}, \mathcal{E}_{\prec}\}$, $\mathcal{R}_{NP} = \mathcal{A}_{NP}$ and $\mathcal{B} = \{(\prec \succ)\}$. Then enumerate each set $T \subseteq \mathcal{A}$ with $|T| \leq |\mathcal{R}_P| = 2$ and test if for each T, either $T \subseteq \mathcal{S}_{\prec}$, $T \subseteq \mathcal{E}_{\prec}$, or $D \subseteq \mathcal{C}_{\mathcal{A}}(T \cup \mathcal{B})$ for some $D \in \mathcal{A}_{NP}$. There are $\approx 3.4 \cdot 10^7$ such subsets. The test succeeds for all T, and the result follows from Theorem 3.5. □

Since both of these algebras also contain the relations $(\equiv), (\prec), (\succ), (\equiv \prec), (\equiv \succ)$, these are the only tractable algebras capable of expressing sequentiality.

In fact, when enumerating subsets in Proposition 3.7, Proposition 3.9 and Proposition 3.10, it is possible to optimise by stopping at subsets known to be NP-complete (those in $\mathcal{A}_{NP}$); sometimes with a factor thirty.

4 Discussion

It is appropriate to indicate the applicability of this method to further classify tractability in $\mathcal{A}$. Therefore, consider the task of classifying all tractable algebras containing the basic relation (s). There are nine known maximal tractable algebras containing this relation. Thus, we have to enumerate all subsets of an 8192-element set having nine or fewer elements. This amounts to $4.6 \cdot 10^{29}$ subsets, making this task more difficult by a factor of 10^{19}, which is clearly impossible using today's computers.

For the full classification, we certainly need methods that combine theoretical studies of the structure of $\mathcal{A}$ with brute-force computer methods, similar to how the four-colour theorem was proved [Appel and Haken, 1976].

5 Conclusion

We have partially classified tractability of reasoning in Allen's interval algebra, with the result that any yet unknown tractable subclass can contain at most the basic relations $(\equiv)$, (b), $(b^{\smile})$, where $b \in \{\mathsf{d}, \mathsf{o}, \mathsf{s}, \mathsf{f}\}$. This means that for specifying complete knowledge about temporal relations, there is no hope of finding more expressive and yet tractable subclasses than those known today. Furthermore, we completely characterise the set of tractable subclasses which can express the notion of *sequentiality*, which is useful in many AI contexts.

Appendix

Here the intractability proofs needed for the proof of Proposition 3.2 are collected.

A.1 Model Transformations

Definition A.1 (Subsets Δ_0 and Δ_4) Let $R = (\mathsf{d}\ \mathsf{d}^{\smile}\ \mathsf{o}\ \mathsf{o}^{\smile})$ as in Definition 3.1, and define $\Delta_0 = \{(\prec \succ), R \cup (\equiv\ \mathsf{m}\ \mathsf{m}^{\smile}\ \mathsf{s}\ \mathsf{s}^{\smile}\ \mathsf{f}\ \mathsf{f}^{\smile})\}$, and $\Delta_4 = \{(\prec \succ), R \cup (\equiv\ \prec\ \mathsf{m}\ \mathsf{m}^{\smile}\ \mathsf{s}\ \mathsf{s}^{\smile}\ \mathsf{f}\ \mathsf{f}^{\smile})\}$. □

Proposition A.2 $\mathcal{A}$-SAT(Δ_0) and $\mathcal{A}$-SAT(Δ_4) are NP-complete.
Proof: See Golumbic and Shamir [1993]. □

The NP-completeness results of Proposition A.2 can be exteded considerably by techniques introduced next.

Our main vehicle for showing intractability of different subclasses is that of *model transformations*. It is a method for transforming a solution of one problem to a solution of a related problem. The concept of model transformation and related results were introduced in the context of temporal reasoning in Jonsson *et al.* [1996].

Definition A.3 (Model transformation) A *model transformation* is a mapping on $\mathcal{A}$-interpretations. □

Next, a way to describe such transformations.

Definition A.4 (Model transformation description) Let T be a model transformation. A function $f_T : \mathbf{B} \rightarrow 2^{\mathbf{B}}$ is a *description* of T iff for arbitrary $\mathcal{A}$-interpretations $\Im$, the following holds: if $b \in \mathbf{B}$ and $I(b)J$ under $\Im$ then $I(f_T(b))J$ under $T(\Im)$. A description f_T can be extended to handle disjunctions in the obvious way: $f_T(R) = \bigcup_{r \in R} f_T(r)$. □

We can now provide a result on how model transformations can be used.

Lemma A.5 Let $\mathcal{R} = \{r_1, \ldots, r_n\} \subseteq \mathcal{A}$ and $\mathcal{R}' = \{r'_1, \ldots, r'_n\} \subseteq \mathcal{A}$ be such that $r'_k \subseteq r_k$ for all $1 \leq k \leq n$, and $\mathcal{A}$-SAT($\mathcal{R}$) is NP-complete. If there exists a model transformation T with a description f_T such that $f_T(r_k) \subseteq r'_k$ for every $1 \leq k \leq n$ then $\mathcal{A}$-SAT($\mathcal{R}'$) is NP-complete.
Proof: Almost identical to one of Jonsson *et al.* [1996]. □

Before we define a model transformation that we will use later on, we need an auxiliary definition (also from Jonsson *et al.* [1996]).

Definition A.6 (Minimal distance) Let $S \subseteq \mathbb{R}$ be finite. The *minimal distance in* S, MD(S), is defined as $\min\{x - y | x, y \in S \wedge x > y\}$. □

Observe that $|S| \geq 2$ in order to make MD(S) defined. This is no problem, since we are working with intervals. For all such S, MD(S) > 0. The definition of minimal distance can be extended to $\mathcal{A}$-interpretations in the following way: Let $\Im$ be an $\mathcal{A}$-interpretation that assigns values to a set of interval variables $\mathcal{I}$, and set $\mathrm{MD}(\Im) = \mathrm{MD}(\{\Im(I^-), \Im(I^+) | I \in \mathcal{I}\})$.

A concrete model transformation follows.

Definition A.7 (Transformation T, description f) Define the model transformation T on $\mathcal{A}$-interpretations assigning values to interval variables $I_1, \ldots, I_n$ as follows. $T(\Im) = \Im'$, where $\Im'$ is obtained from $\Im$ by first setting $\epsilon = \mathrm{MD}(\Im)/(n+1)$ and defining $\Im'(I_i) = \langle \Im(I_i^-) - i\epsilon, \Im(I_i^+) + i\epsilon \rangle$.

Then define $f(b)$ for $r \in \mathbf{B}$ as $f(\equiv) = \{\mathsf{d}, \mathsf{d}^\smile, \mathsf{o}, \mathsf{o}^\smile\}$, $f(\prec) = \{\prec\}$, $f(\succ) = \{\succ\}$, $f(\mathsf{d}) = \{\mathsf{d}\}$, $f(\mathsf{d}^\smile) = \{\mathsf{d}^\smile\}$, $f(\mathsf{o}) = \{\mathsf{o}\}$, $f(\mathsf{m}) = \{\mathsf{o}\}$, $f(\mathsf{m}^\smile) = \{\mathsf{o}^\smile\}$, $f(\mathsf{s}) = \{\mathsf{d}, \mathsf{o}\}$, $f(\mathsf{s}^\smile) = \{\mathsf{d}^\smile, \mathsf{o}^\smile\}$, $f(\mathsf{f}) = \{\mathsf{d}, \mathsf{o}^\smile\}$, $f(\mathsf{f}^\smile) = \{\mathsf{d}^\smile, \mathsf{o}\}$. □

Thus T decreases starting points and increases ending points of intervals, and does this differently for every interval. Now $f(b)$ represents what can happen with the basic relation b when the transformation T is applied.

Proposition A.8 f is a description of T.
Proof: Obvious from the definitions. □

We can now extend the results of Proposition A.2.

Theorem A.9 For any $A \in \mathcal{A}_0 \cup \mathcal{A}_4$ of Definition 3.1, $\mathcal{A}$-SAT(A) is NP-complete.
Proof: Let $r_1 = (\prec\ \succ)$ and $r_2 = R \cup (\equiv\ \mathsf{m}\ \mathsf{m}^\smile\ \mathsf{s}\ \mathsf{s}^\smile\ \mathsf{f}\ \mathsf{f}^\smile)$, where $R = (\mathsf{d}\ \mathsf{d}^\smile\ \mathsf{o}\ \mathsf{o}^\smile)$. Thus $\mathcal{R} = \{r_1, r_2\} = \Delta_0$ and is NP-complete. Take $A \in \mathcal{A}_0$. Now $A = \{r'_1, r'_2\}$ for $r'_1 = (\prec\ \succ)$ and $R \subseteq r'_2 \subseteq R \cup (\equiv\ \mathsf{m}\ \mathsf{m}^\smile\ \mathsf{s}\ \mathsf{s}^\smile\ \mathsf{f}\ \mathsf{f}^\smile)$, and it is obvious that the conditions of Lemma A.5 are satisfied. NP-completeness follows.

Similarly, let $r_1 = (\prec\ \succ)$ and $r_2 = R \cup (\equiv\ \prec\ \mathsf{m}\ \mathsf{m}^\smile\ \mathsf{s}\ \mathsf{s}^\smile\ \mathsf{f}\ \mathsf{f}^\smile)$, where $R = (\mathsf{d}\ \mathsf{d}^\smile\ \mathsf{o}\ \mathsf{o}^\smile)$. Thus $\mathcal{R} = \{r_1, r_2\} = \Delta_4$ and is NP-complete. Take $A \in \mathcal{A}_4$. Now $A = \{r'_1, r'_2\}$ for $r'_1 = (\prec\ \succ)$ and $R \subseteq r'_2 \subseteq R \cup (\equiv\ \prec\ \mathsf{m}\ \mathsf{m}^\smile\ \mathsf{s}\ \mathsf{s}^\smile\ \mathsf{f}\ \mathsf{f}^\smile)$, and it is obvious that the conditions of Lemma A.5 are satisfied. NP-completeness follows. □

A.2 5-Composition

In order to find the last necessary NP-completeness results, we introduce a new operation to the Allen algebra.

Definition A.10 (5-composition) Let $r_1, \ldots, r_5 \in \mathcal{A}$. Define the *5-composition* of $r_1, \ldots, r_5$, denoted *5comp*$(r_1, \ldots, r_5)$, by $I\, \mathit{5comp}(r_1, \ldots, r_5) J \Leftrightarrow \exists K, L.\ I r_1 K,\ K r_2 J,\ I r_3 L,\ L r_4 J,\ K r_5 L$. □

The 5-composition of relations $r_1, \ldots, r_5$ can easily be computed by using Nebel's software for computing satisfiability of networks of Allen relations[1], by constructing a network of four interval variables with relations according to the definition, and computing the entailed relation between two of the variables by choosing the basic relations which are consistent there.

NP-completeness results can be obtained as follows.

Proposition A.11 Let $A \subseteq \mathcal{A}$, and suppose $\mathcal{A}$-SAT($A \cup \{\mathit{5comp}(r_1, \ldots, r_n)\}$) can be shown to be NP-complete, for $r_i \in A$. Then $\mathcal{A}$-SAT(A) is NP-complete.
Proof: Any network expressed using the extended set of relations can be converted to an equivalent one using only relations from A, by the definition of 5-composition. The transformation is obviously polynomial. □

Theorem A.12 $\mathcal{A}$-SAT for the subclasses $\mathcal{N}_3$, $\mathcal{N}_4$ and $\mathcal{N}_5$ are NP-complete.
Proof: Recall the definitions:
$\mathcal{N}_3 = \{(\prec\ \succ), (\mathsf{o}\ \mathsf{o}^\smile)\}$, $\mathcal{N}_4 = \{(\prec\ \succ), (\mathsf{o}\ \mathsf{o}^\smile\ \mathsf{m}\ \mathsf{m}^\smile)\}$ and $\mathcal{N}_5 = \{(\mathsf{m}\ \mathsf{m}^\smile), (\prec\ \succ\ \mathsf{s}\ \mathsf{s}^\smile\ \mathsf{f}\ \mathsf{f}^\smile)\}$. Define $c(r_1, r_2) = \mathit{5comp}(r_1, r_1, r_1, r_1, r_2)$. First, we verify that $r_3 = c((\mathsf{o}\ \mathsf{o}^\smile), (\prec\ \succ)) = (\equiv\ \mathsf{d}\ \mathsf{d}^\smile\ \mathsf{o}\ \mathsf{o}^\smile\ \mathsf{s}\ \mathsf{s}^\smile\ \mathsf{f}\ \mathsf{f}^\smile)$, and we see that $A \subseteq \mathcal{N}_3 \cup \{r_3\}$ for some $A \in$

[1] This software was developed for obtaining the results of Nebel's paper [1996], and can be obtained from Bernhard Nebel.

$\mathcal{A}_0$, and NP-completeness follows. Next, $r_4 = c((\mathsf{o}\ \mathsf{o}^\smile\ \mathsf{m}\ \mathsf{m}^\smile), (\prec\ \succ)) = (\equiv\ \mathsf{d}\ \mathsf{d}^\smile\ \mathsf{o}\ \mathsf{o}^\smile\ \mathsf{s}\ \mathsf{s}^\smile\ \mathsf{f}\ \mathsf{f}^\smile)$, and we see that $A \subseteq \mathcal{N}_4 \cup \{r_4\}$ for some $A \in \mathcal{A}_0$, and NP-completeness follows. Last, $r_5 = c((\mathsf{m}\ \mathsf{m}^\smile), (\prec\ \succ\ \mathsf{s}\ \mathsf{s}^\smile\ \mathsf{f}\ \mathsf{f}^\smile)) = (\equiv\ \mathsf{s}\ \mathsf{s}^\smile\ \mathsf{f}\ \mathsf{f}^\smile)$, and it can be verified using Nebel and Bürckert's software [1993] that $\Delta_0 \subseteq \mathcal{C}_{\mathcal{A}}(\mathcal{N}_5 \cup \{r_5\})$, implying NP-completeness. □

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